Prop Let $(W,S) \longleftrightarrow (\Phi, \Delta)$. Then

W finite $\Longleftrightarrow$ Φ finite

Pf

$\Rightarrow$ W finite

$\Phi \longleftrightarrow$ reflections in W $\Rightarrow$ finite

$\Leftarrow$ Sup Φ finite.

Each $w \in W$ permutes Φ, and permuts are different:

$w_1\beta = w_2\beta \ \forall\beta \Rightarrow \beta = w_1^{-1}w_2\beta \ \forall\beta$

$\Rightarrow w_1^{-1}w_2\alpha \geq 0 \quad \forall\alpha \in \Delta$

$\Rightarrow w_1^{-1}w_2 = e.$

So $|W| \leq \#$ permuts of $\Phi < \infty$.

Our next main goal is to classify all finite Coxeter groups / root systems.

Crystallographic Root Systems

(Φ, Δ) is crystallographic if

$$-2\frac{\langle\alpha,\beta\rangle}{\langle\beta,\beta\rangle} \in \mathbb{Z} \quad \text{for all } \alpha,\beta \in \Phi$$

(write $\beta^\vee = \dfrac{2\beta}{\langle\beta,\beta\rangle} \to \langle\alpha,\beta^\vee\rangle \in \mathbb{Z}$)

[motivation:

(crystallographic root systems) $\longleftrightarrow$ (simply-connected compact Lie groups)]

Ex: A_n: $(e_i - e_j)^\vee = e_i - e_j$

$(e_i - e_j, e_k - e_l) \in \mathbb{Z}$ ✓

Prop (Φ, Δ) root system

(a) If $\langle\alpha, \beta^\vee\rangle \in \mathbb{Z}$ for $\alpha, \beta \in \Delta$ then (Φ, Δ) is crystallographic.

(b) If (Φ, Δ) is cryst. then $\Phi \subseteq \mathbb{Z}\Delta$

Pf (b) is clear.

(a) Sup $\langle \alpha, \beta^\vee \rangle \in \mathbb{Z}$ for $\alpha, \beta \in \Delta$.

Claim: $\Phi \subseteq \mathbb{Z}\Delta$

need $w \cdot \alpha \in \mathbb{Z}\Delta$ for $w \in W, \alpha \in \Delta$

need $\sigma_\beta \alpha \in \mathbb{Z}\Delta$ for $\alpha, \beta \in \Delta$

$$\sigma_\beta \alpha = \alpha - \underbrace{\langle \alpha, \beta^\vee \rangle}_{\mathbb{Z}} \beta \quad \checkmark$$

Now need $\langle \gamma, \delta^\vee \rangle \in \mathbb{Z}$ for $\gamma, \delta \in \Phi$

Let $\delta = w\alpha$, $w \in W$, $\alpha \in \Delta$ $\rightarrow \delta^\vee = \dfrac{2w\alpha}{\langle w\alpha, w\alpha \rangle} = w\alpha^\vee$

So
$$\langle \gamma, \delta^\vee \rangle = \langle \gamma, w\alpha^\vee \rangle$$
$$= \langle \underbrace{w^{-1}\gamma}_{\in\, \mathbb{Z}\Delta}, \alpha^\vee \rangle \in \mathbb{Z} \quad \blacksquare$$

The Cartan matrix of (Φ, Δ) is

$$A = [\langle \alpha, \beta^\vee \rangle]_{\alpha, \beta \in \Delta}$$

Ex For $S_4 = A_3$ it is

	e_1-e_2	e_2-e_3	e_3-e_4
e_1-e_2	2	-1	0
e_2-e_3	-1	2	-1
e_3-e_4	0	-1	2

Note. If I know the Cartan matrix, then I know $\Phi = W\Delta$ since I know

$$\sigma_\alpha(\beta) = \beta - 2\langle \alpha, \beta^\vee \rangle \alpha \quad \alpha, \beta \in \Delta$$

so I know

$\sigma_\alpha(v)$ for $v \in \operatorname{span}\Delta$

and then I know

$w(v)$ for $w \in W$, $v \in \operatorname{span}\Delta$

So I know $W\Delta$.

So classifying crystal. root systems $\updownarrow$ classifying integer Cartan matrices

So let's do this: ⤴

Necessary:
- $\langle \alpha, \alpha^\vee \rangle = 2$
- $\langle \alpha, \beta^\vee \rangle \leq 0$ for $\alpha \neq \beta$
- $\langle \alpha, \beta^\vee \rangle = 0 \iff \langle \alpha^\vee, \beta \rangle = 0$

(Not enough.)