Fundamental Domains

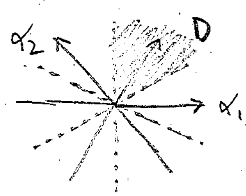
Note. We focus on $\langle , \rangle$ Euclidean, which is possible iff W is finite.
This also works for W infinite with some subtleties.

(W,S) Coxeter system
Π – root system Δ – simple roots

For each root $\alpha \in \Pi^+$

$$H_\alpha = \{\lambda \in V : (\alpha,\lambda) = 0\}$$
$$A_\alpha = \{\lambda \in V : (\alpha,\lambda) > 0\}$$
$$C = \bigcap_{\alpha \in \Pi^+} A_\alpha$$
$$D = \overline{C} = \{\lambda \in V : (\alpha,\lambda) \geq 0 \text{ for all } \alpha \in \Delta\}$$

Ex S_3:



D is a closed convex cone.
It is a fundamental domain for the W-action: every elt of V is conjugate to a unique elt of D.

Prop Every $\lambda \in V$ is $w\mu$ for a unique $\mu \in D$.
(W-conjugate to)

Pf Define poset $<$ on Π^+ by

$\lambda \leq \mu$: $\mu - \lambda$ is a ≥ 0 combin of Δ

Consider those $w\lambda$ with $w\lambda \geq \lambda$ and let μ be maximal among them. (At least λ is in)
Claim: $\mu \in C$.
Otherwise some $(\alpha,\mu) < 0$

$$s_\alpha \mu = \mu - 2(\alpha,\mu)\alpha > \mu$$
↑ also W-conjugate

Unique. Sup $w\lambda = \mu$, $\lambda \neq \mu$ in D, $\ell(w)$ min.
Take $\begin{cases} s \in S \text{ with } \ell(ws) < \ell(w) \\ \alpha \in \Pi^+ \text{ with } w\cdot\alpha < 0 \end{cases}$

$$-w\cdot\alpha > 0 \longrightarrow (-w\alpha,\mu) \geq 0$$
$$0 \geq (w\alpha,\mu) = (\alpha, w^{-1}\mu) = (\alpha,\lambda) \geq 0$$

So $(\alpha,\lambda) = 0$ so $s\lambda = \lambda - 2(\alpha,\lambda)\alpha = \lambda$

$\underbrace{ws}_{\text{shorter}}\lambda = w\lambda = \mu$ $\Rightarrow\Leftarrow$. ∎

Corollary. $w\lambda = \mu$, $\lambda,\mu \in D \Rightarrow \lambda = \mu$
w = prod. of reflections fixing λ