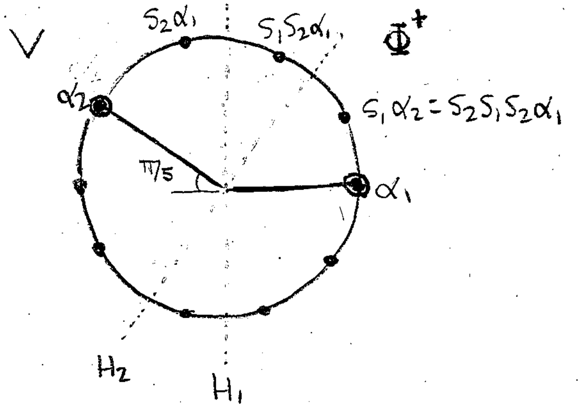
Ex $D_5 = \overset{s_1}{\circ}\overset{}{\underset{5}{\text{—}}}\overset{s_2}{\circ}$ $\qquad \langle \alpha_1, \alpha_2 \rangle = -\cos(\pi/5)$



W = Coxeter group

$S = \{s_1, \dots, s_n\}$ generators

- $\alpha_1, \dots, \alpha_n$ – basis for $\mathbb{R}$-vector space V
- $\sigma_1, \dots, \sigma_n$ – reflections in $GL(V)$
  geom rep'n of W

We say W "acts on" V – it acts on $\alpha_1, \dots, \alpha_n$

Let the <u>root system</u>
$$\Phi = \{ w\alpha_i \mid w \in W,\ 1 \le i \le n \}$$

Elts of Φ are "roots"

$\{\alpha_1, \dots, \alpha_n\}$ are "simple roots".

Say $\alpha \in \Phi$ is positive $(\alpha > 0)$ if
$$\alpha = \sum_{i=1}^{n} c_i \alpha_i \qquad c_i \ge 0$$
and negative if all $c_i \le 0$.

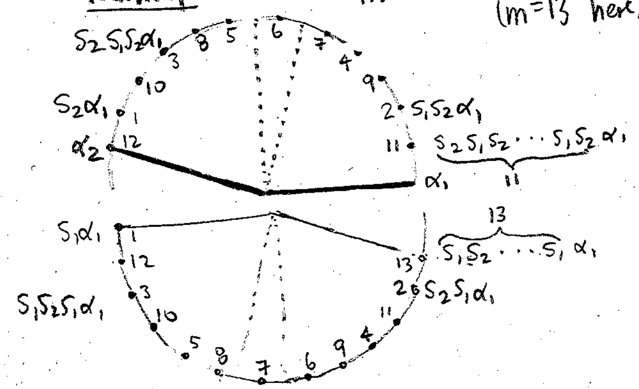
$\Phi^+ = \{\text{positive roots}\}$

$\Phi^- = \{\text{negative roots}\}$

We have $\Phi = \Phi^+ \cup \Phi^-$ because:

$$\boxed{\begin{array}{lll} \textbf{Theorem} \text{ Let } w \in W,\ s_i \in S & & \\ \ell(ws_i) > \ell(w) & \Rightarrow & w\alpha_i > 0 \\ \ell(ws_i) < \ell(w) & \Rightarrow & w\alpha_i < 0 \end{array}}$$

Warmup: $W = \overset{s_1}{\circ}\underset{m}{\text{—}}\overset{s_2}{\circ}$ $\quad$ m odd. ($m = 13$ here) $\quad$ (Exercise – m even)



Φ^+ – must end in s_2

Φ^- – can end in s_1

Now we proceed. First ①$\Rightarrow$② since then
$$\ell(ws_i) < \ell(w) = \ell(ws_is_i) \Rightarrow ws_i\alpha_i > 0$$
$$w(-\alpha_i) > 0$$
$$w\alpha_i < 0.$$

Now need $\ell(ws_i) > \ell(w) \Rightarrow w\alpha_i > 0$

Induct on $\ell(w)$. ($\ell(w)=0$ is obvious)

Take s with $ws < w$.

can't end in s, s_i $\uparrow$ gen. by s, s_i $\downarrow$

Let $J = \{s, s_i\}$ and use $W = W^J W_J$

$\Rightarrow w = w^J w_J$

- $\ell(w) = \ell(w^J) + \ell(w_J)$
- $w^J = \min$ in coset wW_J

- $w\alpha_i = w^J \underbrace{w_J \alpha_i}$

$\rightarrow$ Note $\ell(w_J s_i) > \ell(w_J)$ since

$$\underbrace{ws_i}_{W} = \underbrace{w^J}_{W^J} \underbrace{w_J s_i}_{W_J} \qquad \ell(ws_i) = \ell(w^J) + \ell(w_J s_i)$$

$$\vee$$

$$\ell(w) = \ell(w^J) + \ell(w_J)$$

So by dihedral case

$$w_J \alpha_i = a\alpha_i + b\alpha \qquad a, b \geq 0$$

$\rightarrow$ So $w\alpha_i = w^J(a\alpha_i + b\alpha)$

Is $w^J\alpha_i > 0$? Is $w^J\alpha > 0$?

Note, $\ell(w) > \ell(w^J)$ since $w \notin W^J$ ($\ell(ws) < \ell(w)$)

— By induction, enough to show

$$\ell(w^J s_i) > \ell(w^J) \qquad \ell(w^J s) > \ell(w^J)$$

But these follow since

$$w^J s_i = w \underbrace{w_J^{-1} s_i} \in wW_J$$

$$w^J s \in wW_J$$

$\boxtimes$