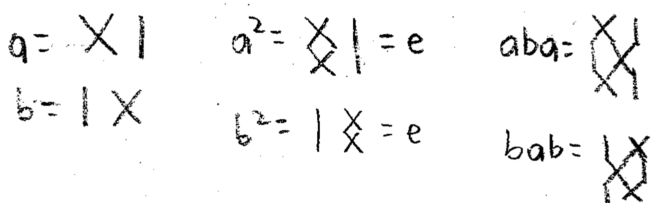
Can shorten: ∘ double letters → leave ababa…

∘ $abab = ba$

$baba = ab$.

So elements: $\{e, a, b, ab, ba, aba\}$ with $aba = bab$

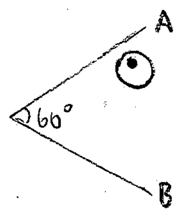
Why the same?



Operation: "stretching wires"

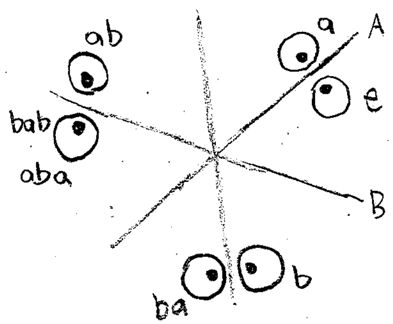
③ Geometry

Kaleidoscope
↓
Six reflections



a = reflect on mirror A

b = " " " B



$a^2 = e$

$b^2 = e$

$aba = bab$

Lecture 2
Jan 28, 08

$\boxed{\text{Coxeter Systems}}$

Coxeter matrix:

$m: S \times S \to \{1, 2, \dots, \infty\}$

$m(s,s') = m(s',s)$

$m(s,s) = 1$

$m(s,s') > 1 \quad s \neq s'$

Coxeter diagram:

Vertices: S

edges:

$s \cdot \quad \cdot s'$ $\quad m(s,s') = 2$

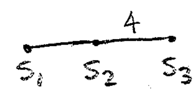
s — s' $\quad m(s,s') = 3$

$s \overset{m}{—} s'$ $\quad m(s,s') = m > 3$

Ex:

$$\begin{array}{c|ccc} & s_1 & s_2 & s_3 \\ \hline s_1 & & & \\ s_2 & & \begin{bmatrix} 1 & 3 & 2 \\ 3 & 1 & 4 \\ 2 & 4 & 1 \end{bmatrix} & \\ s_3 & & & \end{array}$$

→



This defines a Coxeter group:

generators: S → s_1, s_2, s_3

relations: $(ss')^{m(s,s')} = e$ → $s_1^2 = s_2^2 = s_3^2 = (s_1 s_2)^3 = (s_2 s_3)^4 = (s_1 s_3)^2 = e$

Remarks

* No relation when $m(s,s') = \infty$
* $(ss')^n = e \iff \underbrace{ss'ss'\dots}_{n} = \underbrace{s'ss's\dots}_{n}$
* no edge $\overset{\circ}{s} \quad \overset{\circ}{s'}$ means that s, s' commute

(W,S) is called a <u>Coxeter system</u>

$|S|$ is the <u>rank</u> of (W,S)...

Think:

elts of W: words in the alphabet S, regarding

$$\underline{u}\underbrace{(ss'ss'\cdots ss')}_{2m(s,s')}\underline{v} = \underline{u}\ \underline{v}$$

operation: gluing words. $\to$ identity? $\to$ inverse?

Formally, $W \cong F/N$ where

F = free group generated by S

N = normal subgp generated by $\{(ss')^{m(s,s')} \mid s,s' \in S\}$
closure

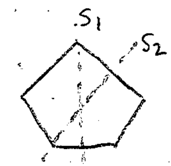
$\boxed{\text{Examples}}$

$s \longrightarrow W = \langle s \mid s^2 = 1\rangle$
$= \{e, s\}$

$s_1 — s_2 \longrightarrow W = \langle s_1, s_2 \mid s_1^2 = s_2^2 = (s_1 s_2)^3 = 1\rangle$
$= S_3$

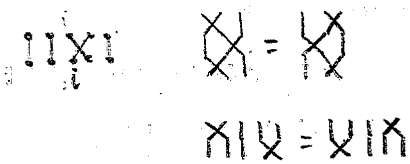
$s_1 \overset{m}{—} s_2 \longrightarrow W = \langle s_1, s_2 \mid s_1^2 = s_2^2 = (s_1 s_2)^m = 1\rangle$

Exercise. $W \cong D_m$ dihedral group



$s_1 — \cdots — s_{n-1} \longrightarrow W = \langle s_1, \ldots, s_{n-1} : s_i^2 = 1,$
$(s_i s_{i+1})^3 = 1,$
$s_i s_j = s_j s_i \ |i-j|>1\rangle$

Exercise: $W \cong S_n$ (Moore)

$s_i =$ 

⚠ A group W can have different presentations as a Coxeter system. (Exercise: D_6)

<u>Important examples</u>

- groups generated by geometric reflections (will say more)
- groups of symmetries of the regular polytopes. (will say more)
- Weyl groups of root systems (will say more) / Lie algebras / Lie groups (will not — project?)