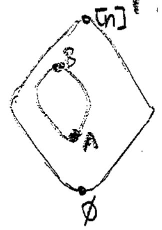
Examples of Eulerian posets

① Boolean poset of subsets of $[n]$



$$\sum_{A\subseteq C\subseteq B} (-1)^{|C|-|A|} = (1+-1)^{|B|-|A|} = 0$$

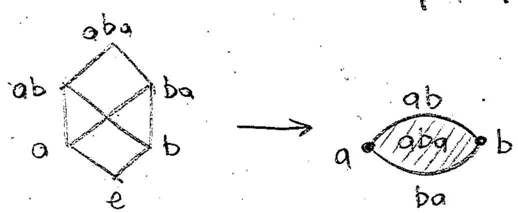
② Face poset of a convex polytope

(By Euler's formula.
Reason: "Euler characteristic" of a ball is 1)

③ Bruhat intervals

↓

Question: Are Bruhat intervals face posets of convex polytopes?



Conj. (Fomin–Shapiro) Bruhat intervals are face posets of "CW-complexes" of balls.

Hersh announced a proof (Dec 07)

3. Weak Order and Reduced Words

The weak order is a (weaker) poset on the elements of a Coxeter group which is useful to study reduced decompositions.

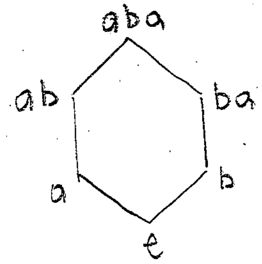
Def $u \leq_R v$ if $v = u s_1 s_2 \cdots s_k$ for some $s_i \in S$ with $\ell(v) = \ell(u)+k$

So in Bruhat order, go up by reflections T
in weak order, go up by simple reflections S

Recall: In Bruhat order $u \to ut$ if $\ell(u)<\ell(ut)$
or equivalently $u \to tu$ if $\ell(u)<\ell(tu)$
$(tu = u(u^{-1}tu))$

Note here! We have defined the right weak order.
We also have a left weak order with all the same properties.
(We focus on right.)

Ex right order on S_3:



Prop

- (reduced words for w) $\longleftrightarrow$ (maximal chains in $[e,w]_R$)

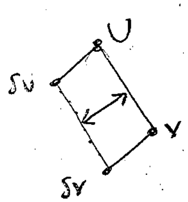
- $u \leq_R v \iff$ there are reduced expressions

  $u = s_1 \cdots s_k$

  $v = s_1 \cdots s_k \cdots s_n$

  PREFIX PROPERTY

- The right order is graded by length. CHAIN PROP.

- 

  If $u > su$, $v > sv$ then: $u > v \Updownarrow su > sv$

  LIFTING, LOWERING PROPERTY

Recall $T_L(w) = \{ t \in T \mid tw < w \}$

$= \{ s_1 \cdots s_i \cdots s_1 \mid 1 \leq i \leq k \}$ if $w = s_1 \cdots s_k$ reduced

distinct

$\boxed{\text{Prop. } u \leq_R v \iff T_L(u) \subseteq T_L(v)}$

$\Rightarrow$: $u = s_1 \cdots s_k$, $v = s_1 \cdots s_k \cdots s_n$ $\rightarrow$ $T_L(u) \subseteq T_L(v)$ by second desc.

$\Leftarrow$: $u = s_1 \cdots s_k$

$T_L(u) = \{ s_1 \cdots s_i \cdots s_1 \mid 1 \leq i \leq k \}$

Claim: v has a reduced expression

$v = s_1 \cdots s_i s_1' \cdots s_j'$ for all $i \leq k$

Induct on i. $i = 0$ ok.

Sup. $v = s_1 \cdots s_i s_1' \cdots s_j'$

Since $s_1 \cdots s_i s_{i+1} s_i \cdots s_1 \in T_L(v)$,

$s_1 \cdots s_i s_{i+1} s_i \cdots s_1 = s_1 \cdots s_i s_1' s_2' \cdots s_a' \cdots s_1' s_i \cdots s_1$

$s_{i+1} = s_1' \cdots s_a' \cdots s_1'$

$s_{i+1} s_1' \cdots s_{a-1}' = s_1' \cdots s_a'$

$v = s_1 \cdots s_i s_{i+1} s_1' \cdots \hat{s_a'} \cdots s_j'$ ✓

For S_n,

$u \leq_R v \iff$ (pairs out of order in u) $\subseteq$ (pairs out of order in v)

$\iff$ I can obtain v from u by switching adjacent $\_\, i\, j\, \_ \rightarrow \_\, j\, i\, \_$, $i < j$

Exercise. Bruhat order = 1-skeleton of "permutohedron".