Pf

First prove that $\{\tau_{ij}\}$ generates in the easier case <u>when the g_i are monomials.</u>

Suppose $\sum_{\ell=1}^{m} a_\ell g_\ell = 0 \qquad a_\ell \in R$

For a fixed monomial $m e_v$, the $m e_v$ coeff is

$$(*) \quad \sum_{\ell \in S} (\alpha_\ell n_\ell) g_\ell = 0 \qquad \alpha_\ell \in \mathbb{F} \quad n_\ell g_\ell = m e_v$$

n_ℓ, g_ℓ monomials

We prove this is a combin of τ_{ij} by induction on the number of $\alpha_\ell \neq 0$. 0: ✓ 1: can't be

If $\alpha_i, \alpha_j \neq 0$ then $n_i g_i = n_j g_j = m e_v = n g$

$$\tau_{ij}: \left(\frac{n_i}{n}\right) g_i = \left(\frac{n_j}{n}\right) g_j \quad (= g) \qquad g = \operatorname{lcm}(g_i, g_j)$$

So $(*)$ is $\underbrace{\sum_{\ell \neq i,j} \alpha_\ell n_\ell g_\ell + (\alpha_i + \alpha_j) n_j g_j}_{\text{smaller support, so combin of } \tau\text{s.}} + \alpha_i \tau_{ij}$ ✓

So each $(*)$ is a combin of $\tau_{ij} \Rightarrow$ so is $\sum_{\ell=1}^{m} a_\ell g_\ell = 0$

Now in general:

$$\tau_{ij} = m_{ji} \varepsilon_i - m_{ij} \varepsilon_j - \sum f_v^{(ij)} \varepsilon_v$$

where $m_{ji} \operatorname{in}(g_i) = m_{ij} \operatorname{in}(g_j) >$ everything from $f_v^{(ij)} \varepsilon_v$

so $\boxed{\operatorname{in}(\tau_{ij}) = m_{ji} \varepsilon_i}$ (where $i < j$)

We need:

$\operatorname{in}(\operatorname{Ker} \varphi)$ generated by $\operatorname{in}(\tau_{ij})$

So take a syzygy

$$\tau = \sum f_\ell \varepsilon_\ell \qquad f_\ell \in F \qquad \left(\sum f_\ell \boxed{g_\ell} = 0\right)$$

We need: $\operatorname{in}(\tau)$ gen. by $\operatorname{in}(\tau_{ij}) = m_{ji} \varepsilon_i$

Let $\operatorname{in}(f_\ell \varepsilon_\ell) = n_\ell \varepsilon_\ell$

$$\operatorname{in}(\tau) = \operatorname{in}\Big(\sum_{\substack{\uparrow \\ \text{no cancellation}}} f_\ell \varepsilon_\ell\Big) = n_i \varepsilon_i \qquad n_i \varepsilon_i > n_\ell \varepsilon_\ell \quad \ell \neq i$$

$$\Downarrow$$

$$n_i \operatorname{in}(g_i) = \operatorname{in}(n_i g_i) \geq \operatorname{in}(n_\ell g_\ell) = n_\ell \operatorname{in}(g_\ell)$$

In $\sum f_\ell g_\ell = 0$, take the part of max degree $n_i \operatorname{in}(g_i)$:

$$\sum_{\ell \in S} n_\ell \operatorname{in}(g_\ell) = 0 \qquad \text{If } \ell \in S \quad n_i \operatorname{in}(g_i) \overset{\text{up to scalar}}{=} n_\ell \operatorname{in}(g_\ell)$$

$$\Rightarrow i \leq \ell$$

So $\sum_{\ell \in S} n_\ell \varepsilon_\ell$ is a syzygy of the monomials $\operatorname{in}(g_\ell)_{(\ell \in S)}$

So it is a combination of $\tau_{jk} = m_{kj} \varepsilon_j - m_{jk} \varepsilon_k$

The terms contributing to $n_i \varepsilon_i$ must be

$$\tau_{ik} = m_{ki} \varepsilon_i - m_{ik} \varepsilon_k \qquad (k > i)$$

So $n_i \varepsilon_i = \operatorname{in}(\tau)$ is generated by the $m_{ki} \varepsilon_i = \operatorname{in}(\tau_{ik})$ $(k>i)$

∎

This allows us to get

- the syzygies
- the syzygies of the syzygies

⋮

Ex. $R = \mathbb{F}[x,y,z]$ lex, $x>y>z$

$M = R/\langle x,y,z\rangle$ TOM (term over monomial)

- M is generated by $a_1 = 1$.

Syzygies: $x a_1 = 0 \qquad y a_1 = 0 \qquad z a_1 = 0$

$b_1 = x \qquad b_2 = y \qquad b_3 = z$

- $\mathrm{Syz}^1(M)$ generated by x, y, z in R

Syzygies: $\underbrace{y b_1 - x b_2 = 0}_{\tau_{12}} \qquad \underbrace{z b_1 - x b_3 = 0}_{\tau_{13}} \qquad \underbrace{z b_2 - y b_3 = 0}_{\tau_{23}}$

$$c_1 = \begin{bmatrix} y \\ -x \\ 0 \end{bmatrix} \qquad c_2 = \begin{bmatrix} z \\ 0 \\ -x \end{bmatrix} \qquad c_3 = \begin{bmatrix} 0 \\ z \\ -y \end{bmatrix}$$

- $\mathrm{Syz}^2(M)$ generated by $\begin{bmatrix} y \\ -x \\ 0 \end{bmatrix}, \begin{bmatrix} z \\ 0 \\ -x \end{bmatrix}, \begin{bmatrix} 0 \\ z \\ -y \end{bmatrix}$ in R^3

Syzygies: $\underbrace{z c_1 - y c_2 = -x c_3}_{\tau_{12}} \qquad \underbrace{0}_{\tau_{13}} \qquad \underbrace{0}_{\tau_{23}}$

$$d_1 = \begin{bmatrix} z \\ -y \\ x \end{bmatrix}$$

- $\mathrm{Syz}^3(M)$ generated by $\begin{bmatrix} z \\ -y \\ x \end{bmatrix}$ in R^3

Syzygies: 0 in R^1

In symbols:

$\mathrm{Syz}^2(M)$ = $\mathrm{Im}\,\partial_2$ = combins of cols of; = $\ker \partial_1$ = relations in here

$$0 \longrightarrow R^1 \xrightarrow[\partial_3]{\begin{bmatrix} z \\ -y \\ x \end{bmatrix}} R^3 \xrightarrow[\partial_2]{\begin{bmatrix} y & z & 0 \\ -x & 0 & z \\ 0 & -x & -y \end{bmatrix}} R^3 \xrightarrow[\partial_1]{[x\ y\ z]} R^1 \xrightarrow{\epsilon} M \longrightarrow 0$$

Def A $\left\{\begin{matrix}\text{complex} \\ \text{exact sequence}\end{matrix}\right\}$ of R-modules is a sequence of modules and homomorphisms

$$\cdots \longrightarrow M_{n+1} \xrightarrow{f_{n+1}} M_n \xrightarrow{f_n} M_{n-1} \longrightarrow \cdots$$

such that for all n $\left\{\begin{matrix} f_n \circ f_{n+1} = 0 \\ \ker f_n = \mathrm{Im} f_{n+1}\end{matrix}\right\}$

A short exact sequence is

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

so
- f injective
- $\ker g = \mathrm{im} f$
- g surjective

Def A <u>free resolution</u> of an R-module M is an exact sequence

$$\cdots \longrightarrow R^{b_2} \longrightarrow R^{b_1} \longrightarrow R^{b_0} \longrightarrow M \longrightarrow 0$$

of free modules "resolving" M.

<u>Hilbert's syzygy theorem</u>. Every finitely generated module over $R = \mathbb{F}[x_1, \dots, x_n]$ has a free resolution of length n:

$$0 \to R^{b_n} \to \cdots \to R^{b_0} \to M \to 0$$

Applicability: Many properties of modules are easy for free modules, and behave along exact seqs.

<u>Fact</u> There is a unique (up to $\cong$) "minimal" free resolution. b_i = i-th "Betti number"
$\mathrm{Im}\,\varphi_i$ = i-th "syzygy module"