In fact, we might want the R-linear relations between elts of R^n, not just R ("linear algebra over R")

$$a\begin{bmatrix} y^3-x^2y \\ x^3-x^2y \end{bmatrix} + b\begin{bmatrix} -xy-y^2 \\ x^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Def.

An <u>R-module M</u> is "a vector space over R":

- an abelian group M, and
- an action $R\times M \to M$ of R on M such that
  - $(r+s)m = rm+sm$
  - $(rs)m = r(sm)$
  - $r(m+n) = rm+rn$
  - $1m = m$

Ex:

- $R = \mathbb{F}$ (field) $\quad M = \mathbb{F}$-vector space
- $R =$ ring $\quad M = R^n \quad r\begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix} = \begin{bmatrix} rr_1 \\ \vdots \\ rr_n \end{bmatrix}$
- $R =$ ring $\quad M = I$ ideal $\quad r\cdot i \in I$
- $R =$ ring $\quad M = R/I \quad r(s+I) = rs+I$

A subset $A \subset M$ <u>generates</u> M <u>as an R-module</u> if

$$M = \{r_1a_1+\cdots+r_ka_k \mid r_1,\dots,r_k \in R,\ a_1,\dots,a_k \in A\}$$

<u>Ex</u>: $R = \mathbb{F}[x,y] \quad M = \{$polys. of degree $\geq 2\}$

Then $\{x^2, xy, y^2\}$ generate M. (Exercise)

<u>Over fields</u>: (Linear Algebra)

- Every $\mathbb{F}$-vector space is $\cong \mathbb{F}^E$
- Every minl generating set (basis) has the same size $|E|$.
- The elts of a basis have no rels. between them.

<u>Over modules</u>: Not true!

In example above:

- $\{x^2, xy, y^2\}$ is a minl gen. set, but
  $$y(x^2) - x(xy) = 0, \quad y(xy) - x(y^2) = 0.$$
- $M \ncong R^3$
- The minl. gen. sets don't need to have the same size if R is complicated enough. (Doesn't satisfy the "invariant basis number" condition.)

An R-module M is <u>free</u> if it has a free basis E:

- E generates M
- E satisfies no relations $r_1e_1+\cdots+r_ne_n=0 \quad r_i\in R,\ e_i \in E$

So $M \cong R^E = \bigoplus_{e\in E} Re$

We would like to also study relations over modules (such as R^n) - so we develop Gröbner bases over free modules.

Gröbner bases over free modules

$R = \mathbb{F}[x_1, \dots, x_r]$

$F = R^n = Re_1 \oplus \dots \oplus Re_n$ free R-module.

monomial: $x_1^{a_1} \cdots x_r^{a_r} e_i = x^a e_i = m e_i$

term: $a x_1^{a_1} \cdots x_r^{a_r} e_i$ $(a \in \mathbb{F})$

Monomial order: a total order on the monomials of F s.t.

- $m_i e_i > m_j e_j \Rightarrow m m_i e_i > m m_j e_j$
- $m e_i > e_i$

(Simplest: $\square\, e_1 > \square\, e_2 > \dots > \square\, e_n$

within each $\square$ use some monomial order on R)

A Gröbner basis for a submodule $M \subset F$ is a set $\{g_1, \dots, g_m\}$ of M such that $in(M)$ is generated by $in(g_1), \dots, in(g_m)$.

Long division: If $f, g_1, \dots, g_t \in F$ one can write

$$f = \sum_{i=1}^{t} a_i g_i + r \qquad a_i \in R \quad r \in F$$

where
- $in(f) \geq in(a_i g_i)$
- no monomial of r is in $(in(g_1), \dots, in(g_t))$

$$S(g_i, g_j) = \begin{cases} m_{ij} g_i - m_{ji} g_j & \text{if } in(g_i), in(g_j) \text{ involve the same } e_i \\ 0 & \text{otherwise} \end{cases}$$

Buchberger's criterion and algorithm still hold.

Suppose $g_1, \dots, g_m$ are a Gröbner basis. Their syzygy module is

$$\{(a_1, \dots, a_m) \in R^m \mid a_1 g_1 + \dots + a_m g_m = 0\} \subset R^m$$

It is the kernel of

$$\varphi: R^m \longrightarrow M \qquad R^m = \bigoplus_{i=1}^{m} R\varepsilon_i.$$
$$\varepsilon_i \longmapsto g_i$$

When $S(g_i, g_j) = m_{ji} g_i - m_{ij} g_j = \sum_u f_u^{(ij)} g_u$, let

$$\tau_{ij} = m_{ji}\varepsilon_i - m_{ij}\varepsilon_j - \sum f_u^{(ij)} \varepsilon_u$$

Theorem. Suppose $\{g_1, \dots, g_m\}$ is a Gröbner basis with respect to the order $<$ on F. Then $\{\tau_{ij}\}_{1 \leq i < j < m}$ generate the syzygy module. Furthermore, they are a Gröbner basis for the syzygy module wrt the order:

$m\varepsilon_u > n\varepsilon_v \iff$
- $in(m g_u) > in(n g_v)$ in F, or
- $in(m g_u) = in(n g_v)$ (up to a scalar) and $u < v$.

In ex $g_1 = x^2$, $g_2 = xy + y^2$, $g_3 = y^3$ lex, $x > y$

$$\tau_{12} = \begin{bmatrix} \boxed{y} \\ -x+y \\ -1 \end{bmatrix} \qquad \tau_{13} = \begin{bmatrix} \boxed{y^3} \\ 0 \\ -x^2 \end{bmatrix} \qquad \tau_{23} = \begin{bmatrix} 0 \\ \boxed{y^2} \\ -x-y \end{bmatrix}$$

$$in(\tau_{12}) = \begin{bmatrix} y \\ 0 \\ 0 \end{bmatrix} \qquad in\, \tau_{13} = \begin{bmatrix} y^3 \\ 0 \\ 0 \end{bmatrix} \qquad in\, \tau_{23} = \begin{bmatrix} 0 \\ y^2 \\ 0 \end{bmatrix}$$