Application: Computing Syzygies (Eisenbud, 15.4)

Q: Given polynomials $g_1, \dots, g_m$, find the syzygies (linear rels with polyn. coeffs) between them

Ex: $g_1 = x^2$
$g_2 = xy + y^2$

① This is easier if $G = \{g_1, \dots, g_m\}$ is a G.b. for $I = \langle g_1, \dots, g_m \rangle \subset \Bbbk[x_1, \dots, x_n] = R$

② If we can do it for G.b., we can do it for any set.

① Recall Buchberger's criterion – it tells us that

$$S(g_i, g_j) = m_{ij} g_i - m_{ji} g_j \equiv 0 \pmod G \qquad m_{ij} = \frac{\operatorname{lcm}(\operatorname{in} g_i, \operatorname{in} g_j)}{\operatorname{in} g_i}$$

so we get

$$m_{ij} g_i - m_{ji} g_j = \sum_u f_u^{(ij)} g_u$$

So

$$\tau_{ij} = m_{ij} e_i - m_{ji} e_j - \sum_u f_u^{(ij)} e_u$$

is an element of the module of syzygies

$$\{(a_1, \dots, a_m) \mid a_i \in R \ \sum_{i=1}^m a_i g_i = 0\} \subset R^m = \bigoplus_{i=1}^m R e_i$$

Theorem (Schreyer). If $G = \{g_1, \dots, g_m\}$ is a Gröbner basis for I in $R = \Bbbk[x_1, \dots, x_n]$ wrt a term order $<$, then τ_{ij} generate the module of syzygies of G.

Furthermore, ... (later)

In example, say lex $x > y$

$g_1 = x^2$
$g_2 = xy + y^2$

Now

$$S(g_1, g_2) = y g_1 - x g_2 = -xy^2$$
$$= -y g_2 + y^3 \not\equiv 0 \pmod{g_1, g_2}$$

so let
$g_3 = y^3$

$\{x^2, xy+y^2, y^3\}$ G.b.

Then

$$S(g_1, g_3) = y^3 g_1 - x^2 g_3 = 0$$
$$S(g_2, g_3) = y^2 g_2 - x g_3 = y^4 = y g_3$$

So these syzygies generate:

$$\begin{cases} y e_1 + (-x+y) e_2 - e_3 \\ y^3 e_1 \qquad\qquad - x^2 e_3 \\ \qquad\qquad y^2 e_2 - (x+y) e_3 \end{cases}$$

② If you didn't have a G.b. $\{g_1, \dots, g_m\}$ extend to one $\{g_1, \dots, g_m, g_{m+1}, \dots, g_t\}$, whose rels. we compute. Each new g_i appears in an old $S(g_a, g_b)$ for the first time, so solve for e_i in τ_{ab} and sub into the remaining τ_{cd}'s:

$$e_3 = y e_1 + (-x+y) e_2 \Rightarrow \begin{cases} (y^3 - x^2 y) e_1 + (x^3 - x^2 y) e_2 \\ -(xy+y^2) e_1 + x^2 e_2 \end{cases}$$

In fact $-(xy+y^2) e_1 + x^2 e_2$ generates the syzygies of $\{g_1, g_2\}$

In general, changing the set of generators also changes the module of syzygies, but it does so predictably, since it is easy to see the effect of adding or removing a generator.