Initial ideals of lattice ideals

$A = \mathbb{Z}^2$

$Q = \mathbb{N}\{(3,0),(1,1),(0,2)\}$

$L = \mathbb{Z}(2,-6,3)$ relations over $\mathbb{Z}$

$I_L = \langle x^2y^3 - z^6 \rangle$

$\mathrm{in}_{(1,1,1)} I_L = \langle z^6 \rangle$

$\mathrm{in}_{(3,3,2)} I = \langle x^2y^3 - z^6 \rangle$

$w \in \mathbb{R}^n_{>0}$ "weight vector" for $\mathbb{F}[x_1 .. x_n]$

A monomial $c_u x^u$ has weight $w \cdot u = w_1u_1 + \dots + w_nu_n$

For $p \in \mathbb{F}[x_1, \dots, x_n]$,

$\mathrm{in}_w(p) =$ sum of w-max terms of p

For an ideal I,

$\mathrm{in}_w(I) = \langle \mathrm{in}_w(p) : p \in I \rangle$

w is generic for I if this is a monomial ideal

- Gröbner basis w.r.t. w: $\mathcal{G} \subset I$ s.t. $\mathrm{in}_w(\mathcal{G}) = \mathrm{in}_w(I)$.
- reduced G.b.: as before

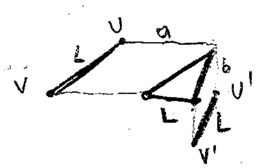
Prop $\{x^u - x^v : \overset{u,v \geq 0}{u - v \in L}\}$ is a G.b. wrt any generic w.

Pf $S(x^u - x^v, x^{u'} - x^{v'})$

$= x^a(x^u - x^v) - x^b(x^{u'} - x^{v'})$

$= x^{b+v'} - x^{a+v}$ ∎

$((b+v') - (a+v) \in L)$



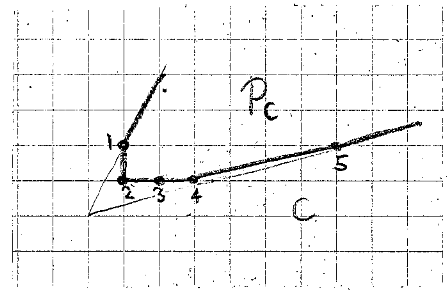
For w generic, $\mathrm{rad}(\mathrm{in}_w(I))$ is the Stanley-Reisner ring of the initial complex $\Delta_w(I)$.

Goal: Describe $\mathrm{in}_w(I)$, $\Delta_w(I)$.

We need more interesting lattice ideals!

C - cone in $\mathbb{Z}^2$.

Hilbert basis:
$(1,2), (1,1), (2,1), (3,1), (7,2)$
(vertices of P_C)

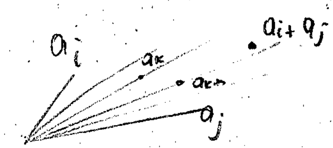


$I_L = \langle \underbrace{x_1x_3 - x_2^3,\ x_2x_4 - x_3^2,\ x_3x_5 - x_4^3}_{}, \underbrace{x_1x_4 - x_2^2x_3,\ x_2x_5 - x_3x_4^2,\ x_1x_5 - x_3^4}_{} \rangle$

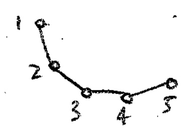
Always $x_{i-1}x_{i+1} - x_i^{\lambda_i}$

- no lattice pts in $\triangle$ $0, a_i, a_{i+1}$
- $\det \boxed{a_i}\boxed{a_{i+1}} = 1$, $\{a_i, a_{i+1}\}$ is a $\mathbb{Z}$-lattice for a_i, a_{i+1}
- $a_{i-1} + a_{i+1} = \lambda_i a_i$

Always $x_ix_j - x_k^{\mu}x_{k+1}^{\nu}$



$\mathrm{in}_{(1,10,100,1000,10000)} I_L = \langle x_1x_3, x_2x_4, x_3x_5, x_1x_4, x_2x_5, x_1x_5 \rangle$

$\Delta_{(1,10,100,1000,10000)} I_L =$  "=" 

A smaller example:

$Q = \mathbb{N}\{1, 5, 10, 25\}$

$I_L = \langle p^5 - n,\ n^2 - d,\ d^2 n - q \rangle$

$W = (1,1,1,1)$

Note: monomial $\leftrightarrow$ some pennies, nickels, dimes, quarters

weight $\leftrightarrow$ number of coins

reduced G.b.: $\langle p^5 - n,\ n^2 - d,\ d^2 n - q,\ d^3 - nq \rangle$
wrt $(1,1,1,1)$

$\mathrm{in}_w(1,1,1,1) = \langle p^5, n^2, d^2 n, d^3 \rangle$

$\uparrow$

minl non-optimal sets of coins.

A monomial $p^i n^j d^k q^l$ is optimal iff $i \le 4 \quad j \le 1 \quad j + k \le 2$

otherwise there are ways of paying the same quantity with fewer coins.

Def Say $u \in \mathbb{N}^n$ is <u>not optimal</u> with respect to $w \in \mathbb{R}^n$ if there is $v \in \mathbb{N}^n$ with $\begin{cases} u - v \in L & (x^u - x^v \in I_L) \\ w \cdot v \le w \cdot u \end{cases}$

$$\boxed{\textbf{Prop}\quad \mathrm{in}_w(I_L) = \mathrm{span}(x^u : u \text{ not } w\text{-optimal})}$$

$\uparrow$ lattice ideals $\qquad\qquad \uparrow$ integer programming

Pf Easy from defs. ∎

Regular triangulations / subdivisions

$\left\{\begin{array}{ll} P = \mathrm{conv}(V) & \text{polytope} \\ h : V \to \mathbb{R} & \text{height on each vertex} \end{array}\right\} \longrightarrow \left(\begin{array}{l}\text{subdivision} \\ P_h \text{ of } P\end{array}\right)$

$P = \mathrm{conv}\{v_1, \dots, v_n\} \subset \mathbb{R}^d$

$\big\downarrow$ "lift"

$= \mathrm{Conv}\{(v_1, h(v_1)), \dots, (v_n, h(v_n))\} \subset \mathbb{R}^{d+1}$

If we look at from below, we see P_h.

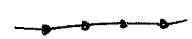
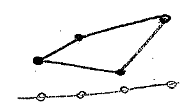
More precisely,

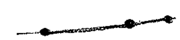
A facet F of $\hat{P}$ is "lower" if the outer normal is $(a, -1)$.

Let $\pi : \mathbb{R}^{d+1} \to \mathbb{R}^d$

$(u_1 \dots u_{d+1}) \mapsto (u_1 \dots u_d)$

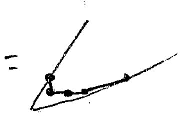
Then $\{\pi(F)\}_{F \text{ lower facet}}$ is a subdiv of P.

Ex. $P =$  $\rightarrow$ $\hat{P} =$ 

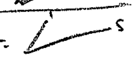
$h = (2, 3, 1, 4)$ $\qquad P_h =$ 

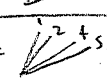
Ex $P =$  $\longrightarrow$ $P_h =$ 

$h = \begin{matrix} 7 & 5 \\ 3 & 10 \end{matrix}$

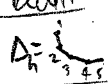
Ex $C =$ 

$h = 1, 10, 100, 1000, 10000 \longrightarrow P_h =$ 

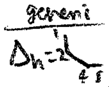
$h = 2, 1, 1, 1, 2 \longrightarrow P_h =$ 

$h = 3, 1, 2, 1, 7 \longrightarrow P_h =$ 

Recall

$\Delta_h =$ 

h not generic

$\Delta_h =$ 

Theorem (w generic)

The initial complex $\Delta_w(I_L)$ "is" the regular subdivision of the cone C corresponding to the lifts w.

Corollary

$\Delta_w(I_L)$ is homeomorphic to a sphere or a ball of dim. $n-\text{rank}(L)-1$.

Pf. See [MS].

What about $\left\{\begin{array}{l}\text{(fine) Hilbert series} \\ \text{(graded) Betti numbers} \\ \text{(graded) free resolutions,}\end{array}\right\}$ for lattice ideals?

Ex

$Q = \mathbb{N}\{(3,0),(1,1),(0,2)\}$

$I_L = \langle x^2z^3 - y^6 \rangle$

Problem: I_L is not homogeneous

Solution: It is if we define $\deg(x_i) = a_i$!

In ex, $\deg x = (3,0)$ $\deg y = (1,1)$ $\deg z = (0,2)$

$$\Rightarrow \deg(x^2z^3) = \deg(y^6) = (6,6)$$

Prop If $Q = \mathbb{N}\{a_1, \dots, a_n\} \subset \mathbb{Z}^d$ then $I_L \subset \mathbb{F}[x_1, \dots, x_n]$ is homog. wrt $\deg x_i = a_i$

Pf Enough for gens $x^u - x^v$, $u-v \in L$

$$\begin{aligned}\deg(x^u) &= u_1a_1 + \dots + u_na_n \\ \deg(x^v) &= \underline{v_1a_1 + \dots + v_na_n} \\ &\quad (u_1-v_1)a_1 + \dots + (u_n-v_n)a_n = 0 \quad \blacksquare\end{aligned}$$

Betti numbers of lattice ideals

Ex 1. $Q = \mathbb{N}\{(3,0),(1,1),(0,2)\}$

$I_L = \langle x^2z^3 - y^6 \rangle$

$$\mathcal{F}: \quad 0 \to \underset{66}{R} \to I_L \to 0 \qquad \text{(boring)}$$

shift

$\boxed{\beta_{0,(6,6)} = 1}$

Ex 2. $Q = \mathbb{N}\{1,5,10,25\}$

$I_L = \langle p^5 - n,\ n^2 - d,\ d^2n - q \rangle$

$$\mathcal{F}: \quad 0 \to \underset{40}{R} \to \underset{\substack{15\\30\\35}}{R^3} \to \underset{\substack{5\\10\\25}}{R^3} \to I_L \to 0$$

$\boxed{\beta_{0,5} = \beta_{0,10} = \beta_{0,25} = 1 \quad \beta_{1,15} = \beta_{1,30} = \beta_{1,35} = 1 \quad \beta_{2,40} = 1}$

Combinatorial / topological formula?

For $b \in \mathbb{N}^d$, let

$$\boxed{\Delta_b = \left\{ I \subseteq [1,\dots,n] \;\middle|\; b - \sum_{i \in I} a_i \in Q \right\}}$$

Prop Δ_b is a simplicial complex.

Pf Sup $I \subseteq J$, $J \in \Delta_b$

Then $b - \sum_{i \in J} a_i \in Q$

$b - \sum_{i \in J} a_i + \sum_{i \in J \setminus I} a_i \in Q$

$b - \sum_{i \in I} a_i \in Q \quad \Rightarrow I \in \Delta_b$. ∎

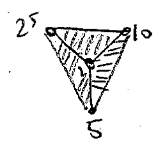
Theorem

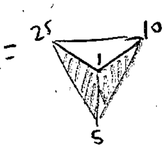
$$\boxed{\beta_{i,b}(I_L) = \dim_{\mathbb{F}} \tilde{H}_i(\Delta_b)}$$

So monomial ideals $\to$ upper Koszul complexes

lattice ideals $\to \Delta_b$

Pf Analogous to monomial case ∎

Ex 2. $\Delta_{37} =$ 

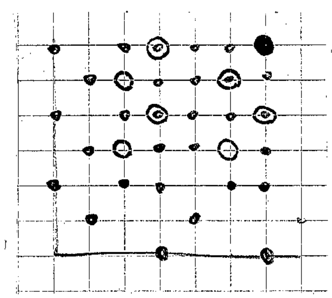
$\Delta_{35} =$ 

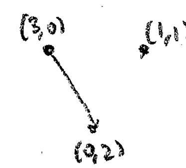
$\beta_{1,35} = 1$

$\Delta_{45} =$ 

hollow

$\beta_{2,45} = 1$

Ex 1 

$\Delta_{(6,6)} =$ 

$\beta_{0,(6,6)} = 1$

From this we can get the Hilbert series with respect to this grading.

Note: $\mathbb{F}[Q] \cong R/I_L$

$$\downarrow$$

Hilbert series $\boxed{\sum_{q \in Q} t^q = \mathrm{Hilb}(R/I_L; t)}$

graded by $\deg x_i = a_i$

Ex 2. $\mathrm{Hilb}(R/I_L; t) =$

$$= \frac{1-(x^5+x^{10}+x^{25})+(x^{15}+x^{30}+x^{35})-x^{40}}{(1-x)(1-x^5)(1-x^{10})(1-x^{25})}$$

$$= \frac{1}{1-x} = \sum_{\substack{q \in \mathbb{Z} \\ \shortparallel \\ Q}} x^q$$

Ex 1 $\mathrm{Hilb}(R/I_L; t) = \dfrac{1-s^6t^6}{(1-s^3)(1-st)(1-t^2)}$

Conclusion

For lattice ideals (or for monomial ideals) we have combinatorial recipes for

- Betti numbers
- Hilbert series
- free resolutions (see [MS]
  - hull resolution
  - Scarf complex