Semigroup rings

semigroup: a set with a binary associative operation

(doesn't nec. have identity, inverses)

The semigroups we care about: - commutative
- with 0
- finitely generated

A - abelian group

$\{a_1, \dots, a_n\} \subset A$

Q - semigp. gen. by $\{a_1 \dots a_n\}$

$Q = \{k_1 a_1 + \dots + k_n a_n \mid k_i \in \mathbb{N}\} = \mathbb{N}\{a_1 \dots a_n\}$

Def The semigroup ring $\mathbb{F}[Q]$ of Q is:

$$\mathbb{F}[Q] = \{\sum_{i=1}^{m} \lambda_i t^{b_i} \mid \lambda_i \in \mathbb{F},\ b_i \in Q\}$$

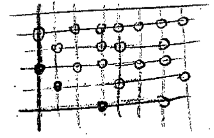
with multiplication $t^a t^b = t^{a+b}$. (t^{a_i} generate as a ring)
(t^a ($a \in Q$) as a vector sp)

Let $\varphi: \mathbb{Z}^n \longrightarrow A$
$e_i \mapsto a_i$

and let $L = \operatorname{Ker} \varphi$, a lattice (discrete subgp) in $\mathbb{Z}^n$.

Def The lattice ideal $I_L \subseteq \mathbb{F}[x_1 \dots x_n] = R$ of L is $I_L = \langle x^u - x^v \mid u, v \in \mathbb{N}^n,\ u - v \in L \rangle$

Ex $\begin{cases} A = \mathbb{Z}^2 \\ Q = \mathbb{N}\{(3,0), (1,1), (0,2)\} \end{cases}$



- $\mathbb{F}[Q] = \mathbb{F}\{t^a : a \in Q\} = \mathbb{F}\{t_1^{3a+b} t_2^{b+2c} : a, b, c \in \mathbb{N}\}$

$\varphi: \mathbb{Z}^3 \longrightarrow A$
$e_1 \mapsto (3,0)$
$e_2 \mapsto (1,1)$
$e_3 \mapsto (0,2)$

$L = \{(a,b,c) \in \mathbb{Z}^3 \mid a(3,0) + b(1,1) + c(0,2) = (0,0)\}$

- $L = \mathbb{Z}\{(2,-6,3)\}$ (relations between a_1, a_2, a_3)
- $I_L = \langle x^2 z^3 - y^6 \rangle \subset \mathbb{F}[x,y,z] = R$

Theorem $\mathbb{F}[Q] \cong R / I_L$

Pf Let $\Phi: R \longrightarrow \mathbb{F}[Q]$
$x_i \mapsto t^{a_i} \qquad 1 \le i \le n$

- $\operatorname{Im} \Phi = \mathbb{F}[Q]$
  Clear since t^{a_i} generate the ring
- $\operatorname{Ker} \Phi = I_L$
  $\supseteq$: $x^u - x^v \in I_L$, $u - v \in L \Rightarrow \varphi(u) = \varphi(v)$

  $\Phi(x^u - x^v) = \Phi(x_1^{u_1} \cdots x_n^{u_n} - x_1^{v_1} \cdots x_n^{v_n})$
  $= t^{a_1 u_1} \cdots t^{a_n u_n} - t^{a_1 v_1} \cdots t^{a_n v_n}$
  $= t^{\varphi(u)} - t^{\varphi(v)} = 0$

  ($\subseteq$: see book.)

Ex $\begin{cases} A = \mathbb{Z}/2\mathbb{Z} \\ Q = \mathbb{N}\{\bar{1},\bar{1},\bar{1}\} \end{cases}$

- $\mathbb{F}[Q] = \mathbb{F}\{t^{\bar{1}}\}$

$$\begin{aligned} \varphi: \mathbb{Z}^3 &\longrightarrow A \\ e_1 &\mapsto \bar{1} \\ e_2 &\mapsto \bar{1} \\ e_3 &\mapsto \bar{1} \end{aligned}$$

- $L = \{(a,b,c) \in \mathbb{Z}^3 \mid a+b+c \text{ even}\}$

- $I_L = \langle x^a y^b z^c - x^d y^e z^f \mid a+b+c+d+e+f \text{ even} \rangle$
$= \langle x^2-1, xy-1, xz-1 \rangle$

Theorem ("Affine semigroups") TFAE:

1. The semigroup Q is <u>affine</u>: it is isomorphic to a subsemigp of some $\mathbb{Z}^d$.
2. The group gen by Q in A is isom. to some $\mathbb{Z}^d$.
3. The semigroup ring $\mathbb{F}[Q]$ is an integral domain
4. The lattice ideal I_L is prime

Pf $1 \Leftrightarrow 2$ clear

$3 \Leftrightarrow 4$ $\mathbb{F}[Q] \cong R/I_L$

$2 \Rightarrow 3$ $\mathbb{F}[Q]$ subring of $\mathbb{F}[\mathbb{Z}^d] = \mathbb{F}[x_1,\dots,x_n,x_1^{-1},\dots,x_n^{-1}]$ which is an I.D.

$\neg 2 \Rightarrow \neg 3$ Sup this subgp has torsion, say $m \cdot a = 0$ $m > 1$.
Let $a = a_1 - a_2$, $a_1, a_2 \in Q \Rightarrow ma_1 = ma_2$
$\Rightarrow t^{a_1} - t^{a_2} \mid t^{ma_1} - t^{ma_2} = 0$. ∎

The point

Now that we understand monomial ideals pretty well (Hilbert series, free resols, Betti numbers), we go to the next simplest family of ideals: lattice ideals.
We will now try to understand these pretty well also – again polyhedral geometry plays a key role.

An aside:

Let n_g be the number of semigroups $Q \subseteq \mathbb{N}$ such that $|\mathbb{N} \setminus Q| = g$.

<u>Conjecture.</u> (Bras-Amorós '08)

$$\lim_{g\to\infty} \frac{n_{g+1}}{n_g} = \frac{1+\sqrt{5}}{2}$$

Known bounds:

- $F_{g+2} - 1 \leq n_g \leq 1 + 3 \cdot 2^{g-3}$ (B-A) $\qquad F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$

- improvements (Elizalde, yesterday) arxiv.org/abs/0905.0489

Another: $Q = \mathbb{N}\{a,b\}$ $\quad (a,b)=1 \quad a,b>0$
$|\mathbb{N}\setminus Q| = ?$ $\quad \max(\mathbb{N}\setminus Q) = ?$

Affine semigroups and cones

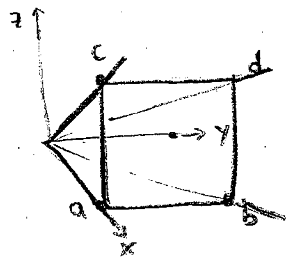
For an affine semigroup

$$Q = \mathbb{N}\{a_1, \dots, a_n\} \qquad a_i \in \mathbb{Z}^d$$

there is a cone

$$\mathbb{R}_{\geq 0} Q = \{\lambda_1 a_1 + \dots + \lambda_n a_n \mid \lambda_i \geq 0\}$$

that is very helpful to us.



Ex $Q = \mathbb{N}\{\underset{a}{(1,0,0)}, \underset{b}{(1,1,0)}, \underset{c}{(1,0,1)}, \underset{d}{(1,1,1)}\}$

$L = \mathbb{Z}\{(1,-1,-1,1)\}$

$I_L = \langle ad - bc \rangle$

$\mathbb{F}[Q] = \mathbb{F}\{t^{(1,0,0)}, t^{(1,1,0)}, t^{(1,0,1)}, t^{(1,1,1)}\} \cong \mathbb{F}[a,b,c,d]/\langle ad-bc\rangle$

Say Q is *pointed* if $a, -a \in Q \Rightarrow a = 0$.

> **Prop** Any pointed affine semigroup Q has a unique finite minl generating set $\mathcal{H}_Q$.

Pf Put a height $h: C \to \mathbb{R}_{\geq 0}$ – we can do it since Q is pointed.
Regard Q as a poset with $a < b \iff b - a \in Q$.
Let $\mathcal{H}_Q$ be the minl elts of the poset – they must be in the ~~generating set~~
Any elt of Q can be written in terms of them by induction on its height ∎

Conversely,

> **Thm** If C is a rational cone in $\mathbb{R}^d$ (rational ineqs.) and A is a subgroup of $\mathbb{Z}^d$, then $C \cap A$ is an affine semigroup. (Gordan)

Pf Might as well assume $A = \mathbb{Z}^d$. We just need $C \cap A$ to be fin. gen.

Let $C = \mathbb{R}_{\geq 0}\{b_1, \dots, b_r\}$ $\quad b_i \in \mathbb{Z}^d$, and

$$\mathrm{Box}(C) = \sum_{i=1}^{r} \lambda_i b_i \mid 0 \leq \lambda_i \leq 1$$

Then $\{b_1, \dots, b_r\} \cup (\mathrm{Box}(C) \cap \mathbb{Z}^d)$ generates. ∎

Def Q affine semigroup in $\mathbb{Z}^d$
A subgroup of $\mathbb{Z}^d$ gen by Q
The *saturation* of Q is $Q_{sat} = (\mathbb{R}_{\geq 0} Q) \cap A$ (cone)

The *Hilbert basis* of Q_{sat} is the unique minl generating set. They are tricky!

Say Q_{sat} has the integer Carathéodory property if any elt. of Q is an $\mathbb{N}$-combin of d elts of the Hilbert basis. (To be expected?)

> **Thm** (Bruns-Gubeladze, 99) There are pointed affine saturated in $\mathbb{Z}^d$ not satisfying ICP, for all $d \geq 6$.

False for $d \leq 3$, open for $d = 4, 5$.