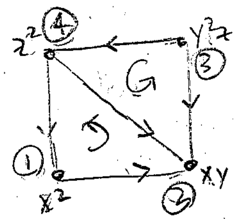
# Generic monomial ideals

## Taylor complex

$\Delta$ simplicial complex with vertex $i$ labeled $m_i$

label face $F$ by $\operatorname{lcm}_{i\in F} m_i = m_F$

The Taylor complex $\mathcal{F}_\Delta$ is the chain complex of $\Delta$ with this grading.




$$\partial(\underset{xy^2z^2}{e_{234}}) = -z\,\underset{xy^2z}{e_{23}} + x\,\underset{y^2z^2}{e_{34}} + y\,\underset{xyz^2}{e_{24}}$$

$$0 \to \underset{\substack{122\\212}}{R^2} \longrightarrow \underset{\substack{121\\022\\202\\210\\112}}{R^5} \longrightarrow \underset{\substack{110\\200\\002\\021}}{R^4} \longrightarrow \underset{0}{I} \longrightarrow 0$$

$\boxed{\textbf{Prop } \mathcal{F}_\Delta \text{ is a free resol. of } I \iff \Delta_{\leq m} \text{ is acyclic for all } m}$

$\boxed{\textbf{Prop } \mathcal{F}_\Delta \text{ is minimal} \iff m_F \neq m_{F-i} \text{ for all } F\in\Delta,\ i\in F}$

(In monomial matrix $\begin{pmatrix}\text{row label}\\ = \\ \text{col label}\end{pmatrix} \Rightarrow$ entry $=0$)

Goal: Every monomial ideal has a <u>simplicial</u> free resol. of length $\leq n$ (not just cellular)

1. generic case
2. reduce any case to generic case

Say $m'$ <u>strictly divides</u> $m$ if

$$x_i \mid m \Rightarrow \deg_{x_i} m' < \deg_{x_i} m$$

Say $\langle m_1 \ldots m_r\rangle$ is <u>generic</u> if

$$\deg_{x_i} m_a = \deg_{x_i} m_b > 0 \Rightarrow \text{some } m_c \text{ strictly divides } \operatorname{lcm}(m_a, m_b)$$

i.e.

$m_a, m_b$ cannot have equal and $>0$ $x_i$-degree $\Leftarrow$  $\in$ Buch $I$

(Recall: strongly generic demands this for <u>all</u> $a, b$)

**Def** The <u>Scarf complex</u> $\Delta_I$ of $I=\langle m_1,\dots,m_r\rangle$ is

$$\Delta_I = \{\sigma\subseteq[r] \mid m_\sigma\neq m_\tau \text{ for all } \sigma\neq\tau\}$$

**Lemma** $\Delta_I$ is a simplicial complex of dim $\leq n-1$

**Pf.** Sup $\sigma\in\Delta_I$
$\tau=\sigma-i\notin\Delta_I$

Then $m_\tau=m_\rho$ for some $\rho\neq\tau$

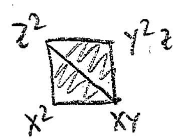
Then $\underbrace{m_{\tau\cup i}}_{\sigma}=m_{\rho\cup i}$ so $\rho\cup i=\sigma$
$\Rightarrow \rho=\sigma-i=\tau$
or $\rho=\sigma$ $\Rightarrow\Leftarrow$

Sup $\sigma\in\Delta_I$ $\dim\sigma\geq n$. Say $\sigma=\{a_1,\dots,a_m\}$

Then one of the $a_i$ doesn't contribute to $m_\sigma=\mathrm{lcm}(m_{a_1},\dots m_{a_m})$

so $m_\sigma=m_{\sigma-a_i}$. ■

**Ex** $I=\langle x^2,xy,z^2,y^2z\rangle \longrightarrow$




**Prop** $\text{edges}(\Delta_I)\subseteq \text{Buch}(I)$

$I$ generic $\Rightarrow \text{edges}(\Delta_I)=\text{Buch}(I)$

The <u>algebraic Scarf complex</u> $\mathcal{F}_{\Delta_I}$ of $I$ is the Taylor complex supported on the Scarf complex $\Delta_I$

**Prop** Every free resol. of $I$ contains the algebraic Scarf complex as a subcomplex

**Pf** Enough for minl free resol.

So start with Taylor resolution on the full simplex, with vertices labelled by $m_1,\dots,m_r$.

Reduce it to a minl one, $\mathcal{F}$.

If $F=\{f_1,\dots,f_k\}\in\Delta_I$ then $m_F$ is unique, so you couldn't have removed it $\Rightarrow F\in\mathcal{F}$. ■

The algebraic Scarf complex $\Delta_I$ is the best possible cellular (simplicial!) resolution, if $I$ is generic

**Thm** Let $\Delta_I$ be the Scarf complex of $I$

- $\Delta_I$ is a subcomplex of $\text{hull}(I)$.
- $I$ generic $\Rightarrow \Delta_I = \text{hull}(I)$

  $\Rightarrow \mathcal{F}_{\Delta_I}$ is a minl free resolution of $R/I$ ....

If see book

**Corollaries**

$I$ generic

- $K(S/I;x) = \sum_{\sigma \in \Delta_I} (-1)^{|\sigma|} m_\sigma$ with no cancellation
- $\beta_i(I) = \sum_a \beta_{i,a}(I)$ is the number of $i$-dim faces of $\Delta_I$

If $I$ is not generic, we can still deform as in 3-D.

**Thm**

$I$ monomial $\quad I = \langle x^{a_1}, \dots, x^{a_m} \rangle$

$\varepsilon$ generic deform. $\quad I_\epsilon = \langle x^{a_1+\varepsilon_1}, \dots, x^{a_m+\varepsilon_m} \rangle$ generic

$\Delta_{I_\varepsilon}$ Scarf complex of $I_\epsilon$

$\Delta_I^\epsilon$ : $\Delta_{I_\varepsilon}$ relabelled to $\varepsilon_i \to 0$

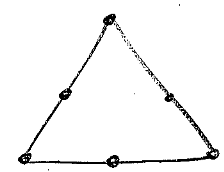
The Taylor complex $\mathcal{F}_{\Delta_I^\varepsilon}$ resolves $R/I$

- length $\leq n$
- generally not minimal

**Ex**

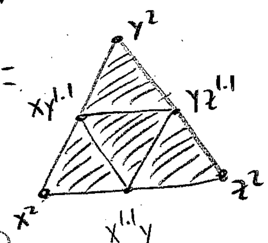
$I = \langle x^2, y^2, z^2, xy, yz, xz \rangle$

$\Delta_I =$



$I_\epsilon = \langle x^2, y^2, z^2, xy^{1.1}, yz^{1.1}, x^{1.1}z \rangle$

$\Delta_{I_\epsilon} =$




$0 \to R^4 \to R^9 \to R^6 \to I \to 0 \to$ (minl since $m_F \neq m_{F-i}$)

$R^4$: 211, 121, 112, 111

$R^9$: 120 etc, 111, 111, 111

$R^6$: 200 etc, 110 etc