Pf by example

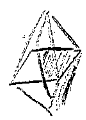
Recall $\beta_{i,b}(I) = \dim_{\mathbb{F}} \left(\mathrm{Tor}_i^R(I, \mathbb{F})\right)_b$

So to get $\beta_{2,1110}$, take

$$\mathcal{F}_x \otimes \mathbb{F} : 0 \to \mathbb{F} \to \mathbb{F}^3 \to \mathbb{F}^{12} \to \mathbb{F}^6 \to \mathbb{F} \to 0$$

$$\begin{array}{ccccc}
1111 & 1111 & \to 1110 & 1100 & 0000 \\
 & [illegible] & \to 1110 & 1010 & \\
 & [illegible] & \to 1110 & 1001 & \\
 & [illegible] & 1101 & 0110 & \\
 & \to [illegible] & 1101 & 0101 & \\
 & [illegible] & 1101 & 0011 & \\
 & [illegible] & 1011 & & \\
 & [illegible] & 1011 & & \\
 & & 1011 & & \\
 & & 0111 & & \\
 & & 0111 & & \\
 & & 0111 & & \\
\end{array}$$

$$(\mathcal{F}_x \otimes \mathbb{F})_{1110} : 0 \to 0 \to \mathbb{F} \to \mathbb{F}^3 \to 0 \to 0 \to 0$$

which correspond to X_b  – not a simplicial complex!

But $X_b = X_{\leq b} \setminus X_{<b}$, so do this:

$\underset{\text{simplicial complexes}}{\uparrow \quad \uparrow}$

$$0 \to \tilde{C}_\bullet(X_{<b}) \to \tilde{C}_\bullet(X_{\leq b}) \to \tilde{C}_\bullet(X_b) \to 0$$

This is an exact sequence of complexes, which gives a <u>long exact sequence for homology</u>:

$$\cdots \to \tilde{H}_i(X_{\leq b}) \xrightarrow{a} \tilde{H}_i(X_b) \xrightarrow{b} \tilde{H}_{i-1}(X_{<b}) \xrightarrow{c} \tilde{H}_{i-1}(X_{\leq b}) \to \cdots$$

Now $x^b \in I \Rightarrow X_{\leq b}$ acyclic

$$\Rightarrow \tilde{H}_{i-1}(X_{<b}) = \ker c = \operatorname{Im} b \cong \tilde{H}_i(X_b) / \underset{\substack{\uparrow \\ 0}}{\ker a}$$

How do we prove this long exact sequence? With the

> **Snake lemma**
>
> If this diagram of vector spaces (abelian groups) commutes
>
> $$\begin{array}{ccccccccc}
(0 \to) & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \to & 0 & \\
 & \downarrow a & & \downarrow b & & \downarrow c & & & \\
0 \to & A' & \to & B' & \to & C' & (\to & 0) &
\end{array}$$
>
> where the rows are exact, then we have an exact sequence
>
> $$(0 \to) \ker a \to \ker b \to \ker c \to \operatorname{coker} a \to \operatorname{coker} b \to \operatorname{coker} c (\to 0)$$

Pf: I'll sketch parts of it live, hopefully with no mistakes. ☒

> long exact sequence for homology:
>
> If $0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$ is an exact sequence of complexes then we have an exact seq:
>
> $$\to H_n(A) \to H_n(B) \to H_n(C) \to H_{n-1}(A) \to H_{n-1}(B) \to H_{n-1}(C) \to$$

Pf. Use the diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \ker a_n & \to & \ker b_n & \to & \ker c_n & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & A_n & \to & B_n & \to & C_n & \to & 0 \\
 & & \downarrow{\scriptstyle a_n} & & \downarrow{\scriptstyle b_n} & & \downarrow{\scriptstyle c_n} & & \\
0 & \to & A_{n-1} & \to & B_{n-1} & \to & C_{n-1} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \operatorname{coker} a_n & \to & \operatorname{coker} b_n & \to & \operatorname{coker} c_n & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

to get

$$
\begin{array}{ccccccccc}
 & & \operatorname{Coker} a_{n+1} & \to & \operatorname{Coker} b_{n+1} & \to & \operatorname{Coker} c_{n+1} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \ker a_{n-1} & \to & \ker b_{n-1} & \to & \ker c_{n-1} & &
\end{array}
$$

and then apply the snake lemma to this.

Some examples of cellular resolutions

0. The resolution of monomial ideals in $\mathbb{F}[x,y,z]$ by a planar map.

1. <u>Taylor resolution</u>

$$I = \langle x^{a_1}, \dots, x^{a_r} \rangle \subset R = \mathbb{F}[x_1, \dots, x_n]$$

Let X be the $(r-1)$-simplex, with vertices labelled by x^{a_i}.

Then $X_{\leq b}$ = simplex of vertices with $a_i \leq b$ is contractible, so $\mathcal{F}_X$ is a free resol., the "Taylor resol." of R/I.

Note: length $= r$ could be really large!
(we can always achieve length $\leq n$)

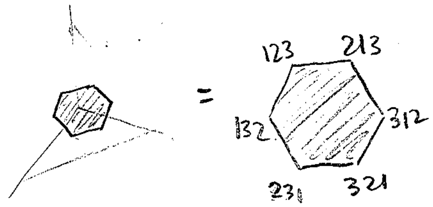
2. <u>Permutation ideals</u>

$$I(u) = \langle x^{\pi(u)} : \pi \in S_n \rangle$$

<u>Ex.</u> $I(123) = \langle xy^2z^3, xy^3z^2, x^2yz^3, x^2y^3z, x^3yz^2, x^3y^2z \rangle$

The <u>permutohedron</u> $P(u)$ is

$$P(u) = \operatorname{conv}(\pi(u) : \pi \in S_n)$$

Ex $P(123) =$ 

Prop The faces of $P(u)$ correspond to the ordered partitions of $[n]$.

Pf The vertices of $P(124578)$ maximizing

$$f(x_1,\dots,x_6) = 5x_1 + 2x_2 + 5x_3 + 2x_5 + 5x_6$$

are those where

$$\{x_1, x_3, x_6\} = \{5,7,8\},\ \{x_2, x_5\} = \{2,4\},\ \{x_4\} = \{1\}$$

The face $P(u)_f$ corresponds to the ordered partition

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Its label is

$$x_1^8\, x_2^4\, x_3^8\, x_4^1\, x_5^4\, x_6^8$$

(Note: vertex 31524 $\updownarrow$ partition 2-4-1-5-3 $\updownarrow$ label $x_1^3 x_2^1 x_3^5 x_4^2 x_5^4$)

Corollary: The combinatorics of $P(u)$ doesn't depend on u.

Is this a free resol?

$P(123)_{\leq 322} =$  $P(123)_{\leq 332} =$ 

Prop: $P(u)_{\leq b}$ is acyclic

Idea: These are the faces of $P(u)$ inside the "orthant" $x \leq b$, and this lets us contract $P(u)_{\leq b}$ to a point

So the permutohedron $P(u)$ supports a cellular resolution of $I(u)$.

3. Hull resolution

The previous construction generalizes.
Let $I \subset \mathbb{F}[x_1 \dots x_n]$ be monomial.
Let

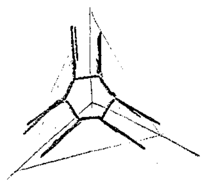
$$P_t = \mathrm{conv}\{\underbrace{(t^{a_1},\dots,t^{a_n})}_{t^a} : x_1^{a_1}\cdots x_n^{a_n} \in I\}$$

Facts:

- P_t is a polyhedron
- $P_t = \mathbb{R}^n_{\geq 0} + \mathrm{conv}(t^a : x^a \in \min(I))$
- The combin. of P_t is indep. of t for $t > (n+1)!$
- The combin. of $(P_t)_{\text{bounded faces}}$ is also indep. of t for $t > (n+1)!$
- The vertices of P_t are t^a for $x^a \in \min(I)$

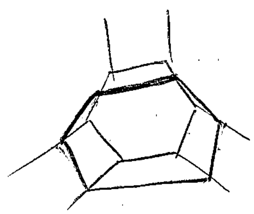
Def The <u>hull complex</u> of I is the polyhedral complex of bounded faces of P_t, with vertices labelled by the minimal generators of I. It is denoted $\mathrm{hull}(I)$.

Ex The hull complex of $I(v)$ is $P(v)$ (as defined above)



Prop $\mathrm{hull}(I)_{\leq b}$ is acyclic.

Pf The vertices of $\mathrm{hull}(I)_{\leq b}$ are those v s.t. $t^{-b} \cdot v < n + \varepsilon$.



Let

$Q = P_t \cap H_{\leq 0}$ (polytope)

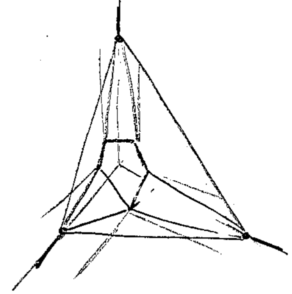
$P = P_t \cap H_{=0}$ (face of P)

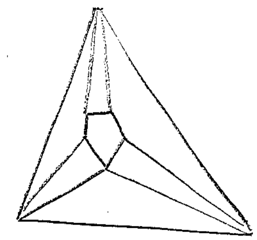
Then $\mathrm{hull}(I)_{\leq b}$ is the set of faces of Q disjoint from face P. That's contractible ∎

Thm $\mathcal{F}_{\mathrm{hull}(I)}$ is a cell. res. of I of length $\leq n$

The artinian case:

If I is artinian ($x_1^{d_1}, \dots, x_n^{d_n} \in I$ for some d_i) the situation is a bit easier to visualize.



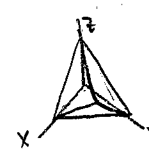


P_t

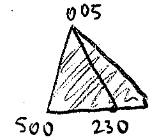
$Q_t = P_t \cap$  $\Delta(I)$

Then:
- Q_t is an n-dimensional polytope
- $\Delta(I)$ is a facet of Q_t
- The bounded faces of P_t (the hull complex) are precisely the faces of Q_t whose inner normal vectors are strictly positive. (All faces of Q_t except $\Delta(I)$ have non-negative inner normal vectors.)
- The hull complex $\mathrm{hull}(I)$ can be seen as a triangulation of the simplex $\Delta(I)$ in dim. $n-1$.

Ex. $I = \langle x^5, y^5, z^5, x^2y^3 \rangle$



$\rightarrow$



$= \mathrm{hull}(I)$