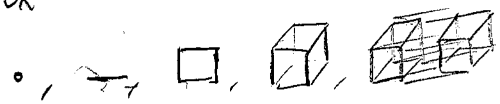
Crash course on polytopes

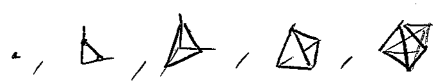
A (convex) polytope is

$$P = \mathrm{Conv}(v_1,\dots,v_n) = \{\lambda_1 v_1 + \dots + \lambda_n v_n \mid 0 \le \lambda_i \le 1,\ \Sigma \lambda_i = 1\} \subset \mathbb{R}^d$$

Ex:



d-cube



d-simplex

Given a linear function w, let

$$P_w = \{p \in P \mid w(p) \text{ is max}\}$$

this is called the w-max face and the faces of P are those arising in this way.

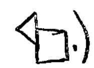
The facets are those of codim 1. A polytope also has an ineq description:

$$P = \{x \in \mathbb{R}^d \mid Ax \le b\}$$

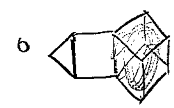
More generally, a polyhedron is anything of that form $\{x \in \mathbb{R}^d \mid Ax \le b\}$, whether bounded or unbounded.

A polyhedral complex X is a finite collection of polytopes (called faces of X) such that:

- If P is a polytope in X, so is every face of P.
- If P, Q are polytopes in X, then $P \cap Q$ is a face of both P and Q.

(No )

Ex: The faces of a polytope



We can orient each face of X in one of two ways.

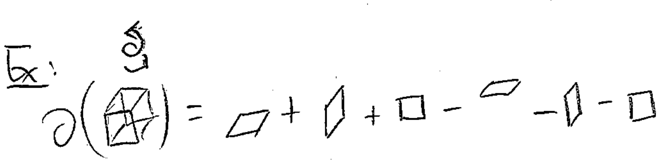
We still have a chain complex

$$0 \to F_d(X) \xrightarrow{\partial_d} \cdots \xrightarrow{\partial_2} F_1(X) \xrightarrow{\partial_1} F_0(X) \xrightarrow{\partial_0} F_{-1}(X) \to 0$$

where

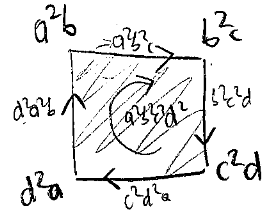
$$\partial F = \sum_{\substack{G \text{ facet} \\ \text{of } F}} \mathrm{sign}(G,F)\, G$$

Ex:



Def A <u>labelled cell cx</u> is a polyhedral cx X with verti v_i labelled by monomials x^{a_i}, and faces ~~F~~ labelled by $\mathrm{lcm}(x^{a_i} : a_i \in F)$

Ex:



$\partial(\square) = \overline{} + | + \underline{} + |$

$$0 \to \underset{2222}{R} \xrightarrow{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}} \underset{\substack{2210\\0221\\1022\\2102}}{R^4} \xrightarrow{\begin{bmatrix}1&0&0&-1\\-1&1&0&0\\0&-1&1&0\\0&0&-1&1\end{bmatrix}} \underset{\substack{2100\\0210\\0021\\1002}}{R^4} \xrightarrow{[1\ 1\ 1\ 1]} \underset{0000}{R^0} \to 0$$

Def The <u>cellular free complex $\mathcal{F}_X$ supported on X</u> is the chain complex of $\mathbb{N}^n$-graded modules with maps given by the boundary maps of X.

(Mini thm: This <u>is</u> a chain complex)

Say it is a <u>cellular resolution</u> if it is <u>acyclic</u> (homology only in deg 0)

Prop The cellular free complex $\mathcal{F}_X$ supported on X is a cellular resolution if and only if the subcomplex $X_{\preceq b}$ is acyclic over $\mathbb{F}$ (trivial homology) for all $b \in \mathbb{N}^n$.

When $\mathcal{F}_X$ is a cellular resolution, it is a free resolution of R/I where

$I = \langle x^{a_i} : i \text{ vertex of } X \rangle$

<u>Pf by example:</u>

In the square to the left, the degree 3321 contributions are:

$$0 \to \underset{\cancel{2222}}{0} \to \underset{\substack{2210\\0221\\\cancel{1022}\\\cancel{2102}}}{R^2} \xrightarrow{\begin{bmatrix}1&0\\-1&1\\0&-1\end{bmatrix}} \underset{\substack{2100\\0210\\0021\\\cancel{1002}}}{R^3} \xrightarrow{[1\ 1\ 1]} \underset{0000}{R^0} \to 0$$

which is the homology chain complex for $X_{\leq 3321}$

(acyclic!)

This is a free resol of R/I since

$\mathrm{Im}\, \partial_0 = I$ $\blacksquare$