Lemma Let I be Borel-fixed

Let $\{m_1, \dots, m_r\}$ be its minl. gen. set

Then every monomial $m \in I$ can be written uniquely as

$$m = m_i m' \quad , \quad \max m_i \le \min m'$$

largest variable in m_i — smallest in m'

Pf Let $u_i = \max m_i$.

Existence: By example, say

$$I = \langle x_1^2, x_1x_2, x_2^3, x_1x_3^3 \rangle$$
$$m = x_1^3 x_2^3 x_3^3$$

Write $m = m_i m'$ in any way.

$$m = (x_1 \underline{x_3}^3)(x_1^2 \underline{x_2}^3)$$

Bad pair? Swap them:

$$m = (x_1 x_2 x_3^2)(x_1^2 x_2^2 x_3)$$

in I (Borel-fixed)

$$= [(x_1x_2)x_3^2](x_1^2x_2^2x_3)$$
$$= (x_1\underline{x_2})(\underline{x_1}^2x_2^2x_3^3)$$

Again:

$$m = (x_1^2)(x_1x_2^3x_3^3) \quad \checkmark$$

This ends since at each stage either $u_i = \max m_i$ or its exponent decreases.

Uniqueness: Not hard. [MS]. ∎

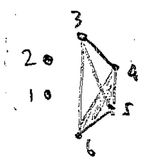
Prop If $I = \langle m_1, \dots, m_r \rangle$ (minl gens) is Borel-fixed,

$$H(I;x) = \sum_{i=1}^{r} \frac{m_i}{\prod_{j=u_i}^{n} (1-x_j)}$$

Pf Easy from lemma ∎

Other nice facts:

- We can compute explicitly the minl. free resolution of Borel-fixed I. ("Eliahou-Kervaire" resolution.)
- We can compute the Betti numbers. The upper Koszul complexes are "shifted":



$$\text{If } \begin{pmatrix} F \in \Gamma \\ f \in F \\ g > f \end{pmatrix} \text{ then } F - f \cup g \in \Gamma.$$

shifted complex

which have easily computed homology:

$\tilde{H}_{-1} = 0$

$\tilde{H}_0 = \mathbb{F}^2$ (1,2)

$\tilde{H}_1 = \mathbb{F}^2$ (34,35)

$$\boxed{\dim_{\mathbb{F}} \tilde{H}_i(\Gamma) = \#\text{ of } i\text{-dim faces } F \in \Gamma \text{ such that } F \cup k \notin \Gamma}$$

(k = max elt)

- Among all ideals with a given Hilbert function H, Macaulay built a Borel-fixed I with the largest number of deg d generators and deg d i-th syzygies (all d, i).