Prop. If I is a monomial ideal in $\mathbb{F}[x_1 \dots x_n]$ then

$$\beta_{i,\mathbf{b}}(I) = \dim_{\mathbb{K}} \widetilde{H}_{i-1}(K^{\mathbf{b}}(I))$$

where $K^{\mathbf{b}}(I) = \{\text{squarefree } \tau \text{ with } x^{\mathbf{b}-\tau} \in I\}$ is the upper Koszul complex of I in degree $\mathbf{b}$.

Pf We need to compute $\mathrm{Tor}_i^R(\mathbb{F}, M) \cong \mathrm{Tor}_i^R(M, \mathbb{F})$

- Resolving M and tensoring by $\mathbb{F}$ gave us the previous result
- Now resolve $\mathbb{F}$ and tensor by M.

We know how to resolve $\mathbb{F} = \mathbb{F}[x_1 \dots x_n]/\langle x_1 \dots x_n \rangle$, we did it in HW2, using the Koszul complex.

$$\mathcal{K}_\bullet : 0 \to R^{\binom{n}{n}} \to R^{\binom{n}{n-1}} \to \dots \to R^{\binom{n}{0}} \to \mathbb{F} \to 0$$

where

$$R^{\binom{n}{i}} = \bigoplus_{\substack{S \subset [n] \\ |S| = i}} R(-\underbrace{11001}_{S})$$

and $\partial_i : R^{\binom{n}{i}} \to R^{\binom{n}{i-1}}$ is:

$$\partial_i(e_{\{a_1, \dots, a_i\}}) = \sum_j (-1)^{j-1} e_{\{a_1, \dots, \hat{a}_j, \dots a_i\}}$$

Ex: $n = 3$

$$\mathcal{K}_\bullet \quad 0 \to R \xrightarrow{\begin{array}{cc} xyz & \\ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} & \begin{array}{c} yz \\ xz \\ xy \end{array} \end{array}} R^3 \xrightarrow{\begin{array}{cc} \begin{array}{ccc} yz & xz & xy \end{array} & \\ \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix} & \begin{array}{c} x \\ y \\ z \end{array} \end{array}} R^3 \xrightarrow{\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \end{array}} R \to \mathbb{F} \to 0$$

$$(\mathcal{K}_\bullet)_{210} : 0 \to 0 \longrightarrow \mathbb{F}^1 \xrightarrow{\begin{array}{cc} xy & \\ \begin{bmatrix} 1 \\ -1 \end{bmatrix} & \begin{array}{c} x \\ y \end{array} \end{array}} \mathbb{F}^2 \xrightarrow{\begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \begin{bmatrix} 1 & 1 \end{bmatrix} \end{array}} \mathbb{F}^1 \to 0 \to 0$$

(vector spaces)

$(\mathcal{K}_\bullet)_{\mathbf{b}}$ is the chain complex of the simplex with vertex set $\mathrm{supp}(\mathbf{b})$. (Homology $= 0$)

Now we tensor $\mathcal{K}_\bullet$ with I:

$$\mathcal{K}_\bullet \otimes I : 0 \to R^{\binom{n}{n}} \otimes I \to \dots \to R^{\binom{n}{0}} \otimes I \to \mathbb{F} \otimes I \to 0$$

$$0 \to I(-1 \dots -1) \to \dots \to \bigoplus_{i=1}^{n} I(-e_i) \to I(0) \to \mathbb{F} \to 0$$

In degree $\mathbf{b}$,

$$(I(-e_{\underbrace{\{a_1 \cdots a_i\}}_{\tau}}))_{\mathbf{b}} = \begin{cases} \mathbb{F} & \text{if } \dfrac{x^{\mathbf{b}}}{x_{a_1} \cdots x_{a_i}} \in I \quad (\tau \in K^{\mathbf{b}}(I)) \\ 0 & \text{ow} \end{cases}$$

So $((\mathcal{K}_\bullet \otimes I)_i)_{\mathbf{b}} = \mathbb{F}^{(i-1)\text{-faces of } K^{\mathbf{b}}(I)}$

and the maps in $(\mathcal{K}_\bullet \otimes I)_{\mathbf{b}}$ are the usual boundary maps.

So $(\mathcal{K}_\bullet \otimes I)_{\mathbf{b}}$ is just the chain complex of $K^{\mathbf{b}}(I)$!

∎

If I is squarefree, there is a simpler version of this.

For $\sigma \in [n]$ let $x^\sigma = \prod_{i \in \sigma} x_i$, $m^\sigma = \langle x_i : i \in \sigma \rangle$.

The <u>Alexander dual</u> of $I = \langle x^{\sigma_1}, \dots, x^{\sigma_r} \rangle$ squarefree

$I^\times = m^{\sigma_1} \cap \dots \cap m^{\sigma_r}$ squarefree

Let $I = I_\Delta$ - then Δ and $\Delta^\times$ are <u>Alexander dual</u>.

$I^\times = I_{\Delta^\times}$

Recall:

Δ - simplicial complex on $[n]$

$$I_\Delta = \langle x^\sigma : \sigma \notin \Delta \rangle = \bigcap_{\tau \in \Delta} m^{[n]-\tau}$$

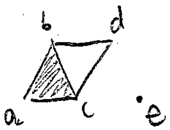
Then

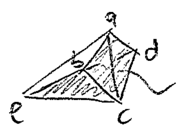
$$I_\Delta^* = \bigcap_{\sigma \notin \Delta} m^\sigma = \langle x^{[n]-\tau} : \tau \in \Delta \rangle$$

So

> **Prop** The Alexander dual Δ^* of Δ is
> $$\Delta^* = \{ [n]-\tau : \tau \notin \Delta \}$$
> ("complements of non-faces")

Ex.

$\Delta =$ 

$\Delta^* =$  hollow tetrahedron

minl non-faces: bcd, ad, ae, be, ce, de

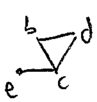
maxl faces: $ae, bce, bcd, acd, abd, abc$

Def Δ - simplicial complex
F - face of Δ

The link of F in Δ is

$$\mathrm{link}_\Delta(F) = \{ G \in \Delta \mid F \cap G = \emptyset,\ F \cup G \in \Delta \}$$

Ex

$\mathrm{link}_\Delta(a) =$  $\quad \mathrm{link}_{\Delta^*}(b) =$  $\quad \mathrm{link}_{\Delta^*}(ab) =$ 

> **Theorem** (Hochster)
> The nonzero Betti numbers of I_Δ and R/I_Δ are all in squarefree degrees σ, and
> $$\beta_{i,\sigma}(I_\Delta) = \beta_{i+1,\sigma}(R/I_\Delta) = \dim_{\mathbb{F}} \tilde{H}_{i-1}(\mathrm{link}_{\Delta^*}(\bar\sigma))$$

squarefree

complement of σ

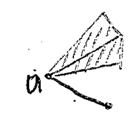
Pf.

- If b is not squarefree, say $b_a \geq 2$

$$\beta_{i,b}(I_\Delta) = \dim_{\mathbb{F}} \tilde{H}_{i-1}(K^b(I))$$

Now, $\tau \in K^b(I) \Longleftrightarrow \tau \cup \{a\} \in K^b(I)$

since $x^{b-\tau} \in I \Longleftrightarrow x^{b-(\tau\cup\{a\})} = \frac{x^{b-\tau}}{x_a} \in I$

So $K^b(I)$ is a cone over a, 
so it is contractible (homotopy equivalent to $\cdot$) and has trivial homology.

- If σ is squarefree, we claim $K^\sigma(I) = \mathrm{link}_{\Delta^*}(\bar\sigma)$.

Step 1: $K^\sigma(I) = \mathrm{link}_{K^1(I_\Delta)}(\bar\sigma)$

Step 2: $K^1(I_\Delta) = \Delta^*$. ▨

> **Prop:** $(\Delta^*)^* = \Delta \qquad (I^*)^* = I$

Pf Easy. (See forum)

Alexander duality and algebra

Recall

$$H(M,x) = \frac{K(M,x)}{\prod_{i=1}^{n}(1-x_i)}$$ ← K-polynomial

for any $\mathbb{N}^n$-graded $\mathbb{F}[x_1 \ldots x_n]$-module.

Theorem (Alexander inversion formula)

$$K(S/I_\Delta ; x) = K(I_{\Delta^*} ; 1-x)$$

Pf. Recall

$$LHS = \sum_{\sigma\in\Delta} \left(\prod_{i\in\sigma} x_i\right)\left(\prod_{j\notin\sigma}(1-x_j)\right)$$

Now

$H(I_{\Delta^*};x) =$ sum of monomials div. by $x^\sigma \in I_{\Delta^*}$ $(\sigma = \bar{\tau}, \tau\in\Delta)$

$$= \sum_{\tau\in\Delta} \prod_{j\notin\tau} \frac{x_j}{1-x_j}$$

$$= \sum_{\tau\in\Delta} \left[\frac{\prod_{i\in\tau}(1-x_i)\prod_{j\notin\tau} x_j}{\prod_{i=1}^{n}(1-x_i)}\right]$$

$\square$

Alexander duality and topology

Thm (Alexander duality)
Let $A \subseteq S^n$ be triangulable.
$\tilde{H}^k(A) = \tilde{H}_{n-k-1}(S^n - A)$

"Originally" this is a duality between homology and cohomology.

The reduced cochain complex of a simplicial cx. Δ is:

$$0 \to \mathbb{F}^{F^*_{-1}(\Delta)} \xrightarrow{\partial^0} \mathbb{F}^{F^*_0(\Delta)} \xrightarrow{\partial^1} \cdots \xrightarrow{\partial^{n-1}} \mathbb{F}^{F^*_{n-1}(\Delta)} \to 0$$

where $\mathbb{F}^{F^*_i(\Delta)}$ is the dual of $\mathbb{F}^{F_i(\Delta)}$, ∂^i is the transpose of ∂_i.

Aside:

- If V is a vector space over $\mathbb{F}$ then $V^* = \{f: V\to\mathbb{F} \mid f \text{ linear}\}$ is the dual v.s.
- If $e_1, \ldots, e_n$ is a basis for V, then $e^1, \ldots, e^n$ is the dual basis of V^* where $e^i(e_j) = \begin{cases} 1 & i=j \\ 0 & \text{ow} \end{cases}$ $(\dim V = \dim V^*)$
- The adjoint of $\varphi: V\to W$ is $\varphi^*: W^*\to V^*$

$$\underbrace{\varphi^*(w^*)}_{V^*}(v) = w^*\varphi(v)$$

with $w^* \in W^*$.

We have, for σ an $(i-1)$-face

$$\partial^i(e^*_\sigma) = \sum_{\substack{j\notin\sigma \\ \sigma\cup j\in\Delta}} \mathrm{sgn}(j, \sigma\cup j)\, e^*_{\sigma\cup j}$$

$(-1)^{(\text{pos. of } j \text{ in } \sigma\cup j)}$

Ex. $\Delta =$ 

$$0 \to \mathbb{F} \xrightarrow{\begin{array}{cc} \emptyset & \\ \begin{bmatrix}1\\1\\1\end{bmatrix} & \begin{matrix}a\\b\\c\end{matrix}\end{array}} \mathbb{F}^3 \xrightarrow{\begin{array}{cc} \begin{matrix}a & b & c\end{matrix} & \\ \begin{bmatrix}1 & -1 & 0\\1 & 0 & -1\\0 & 1 & -1\end{bmatrix} & \begin{matrix}ab\\ac\\bc\end{matrix}\end{array}} \mathbb{F}^3 \xrightarrow{\begin{array}{cc} \begin{matrix}ab & bc & ac\end{matrix} & \\ \begin{bmatrix}1 & -1 & 1\end{bmatrix} & abc\end{array}} \mathbb{F} \to 0$$

$$\emptyset \qquad \begin{matrix}a\\b\\c\end{matrix} \qquad \begin{matrix}ab\\bc\\ac\end{matrix} \qquad abc$$

The cochain complex satisfies $\partial^i \circ \partial^{i+1} = 0$ and the cohomology of Δ is

$$\tilde{H}^i(\Delta) = \ker \partial^{i+1} / \operatorname{Im} \partial^i$$

Since we are working over $\mathbb{F}$,

$$\tilde{H}^i(\Delta) = \tilde{H}_i(\Delta)^{*}$$

The difference arises when we work over other abelian groups.

Anyway,

Theorem (Alexander duality)
$$\tilde{H}_{i-1}(\Delta^*) \cong \tilde{H}^{n-2-i}(\Delta)$$

Corollary (Hochster) The nonzero Betti numbers of I_Δ are for squarefree σ, and
$$\beta_{i,\sigma}(I_\Delta) = \dim \tilde{H}^{|\sigma|-i-2}(\Delta|_\sigma)$$

Proofs:
See Miller-Sturmfels