We have seen that free resolutions sometimes "look like" homology computations.

This is true at a very precise level:

Consider the minl. graded free resolution of a monomial ideal $I \subset R = \mathbb{F}[x_1, \dots x_n]$:

$$0 \to F_\ell \to \cdots \to F_1 \to F_0 \to I \to 0$$

where $F_i = \bigoplus_{a \in \mathbb{N}^n} R(-a)^{\beta_{i,a}}$.

Recall $\beta_{i,a}$ is the <u>i-th Betti number of I in degree a</u>

<u>Def</u> The <u>upper Koszul simplicial complex</u> of I in degree b is

$$K^b(I) = \{\text{squarefree } \tau \mid x^{b-\tau} \in I\}$$

(Check: simp. comp.)

Theorem

$$\boxed{\beta_{i,b}(I) = \dim_{\mathbb{F}} \tilde{H}_{i-1}(K^b(I))} \boxed{= \beta_{i+1,b}(R/I)}$$

We need to do some work to get there.

A handy notation for $\mathbb{N}^n$-graded free resolutions: "monomial matrices"

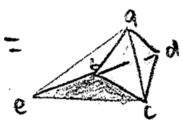
$$\bigoplus_p R(-a_p) \xrightarrow{\begin{array}{c|c} & a_1 \cdots a_p \cdots a_r \\ \hline \begin{matrix} b_1 \\ \vdots \\ b_q \\ \vdots \\ b_s \end{matrix} & \begin{bmatrix} & & \\ & \lambda_{qp} & \\ & & \end{bmatrix} \end{array}} \bigoplus_q R(-b_q)$$

λ_{qp}: shorthand for $\lambda_{qp} x^{b_q - a_p}$

$(\lambda_{qp} = 0$ if $a_p \not\geq b_q)$

Fact: An $\mathbb{N}^n$-graded free resolution is minimal $\iff \lambda_{qp} = 0$ when $a_p = b_q$

<u>Ex</u> The minl free resolution of R/I_Δ for

$\Delta =$ 

$R/I_\Delta = \mathbb{F}[a,b,c,d,e]/\langle abcd, abe, ace, de \rangle$

$$0 \to R \xrightarrow{\begin{bmatrix} -d \\ c \\ -b \\ 0 \end{bmatrix}} R^4 \xrightarrow{\begin{bmatrix} 0 & -ab & -ac & -abc \\ c & d & 0 & 0 \\ -b & 0 & d & 0 \\ 0 & 0 & 0 & e \end{bmatrix}} R^4 \xrightarrow{[de \ abe \ ace \ abcd]} R \longrightarrow R/I_\Delta \to 0$$

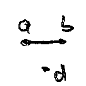
which we write in monomial matrix notation as

$$\cdots \to \underset{\begin{matrix} 11101 \\ 11011 \\ 10111 \\ 11111 \end{matrix}}{R^4} \xrightarrow{\begin{array}{c|c} & abce \ \ abde \ \ acde \ \ abcde \\ \hline \begin{matrix} de \\ abe \\ ace \\ abcd \end{matrix} & \begin{bmatrix} 0 & -1 & -1 & -1 \\ 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}} \underset{\begin{matrix} 00011 \\ 11001 \\ 10101 \\ 11110 \end{matrix}}{R^4} \longrightarrow \cdots$$

$b = 11011$

$i = 1$

$$\beta_{2,b}(R/I) = \dim \tilde{H}_0(K^b(I))$$

$K^b(I)$: $\begin{Bmatrix} \emptyset \\ a, b, d \\ ab \end{Bmatrix}$ 

$\tilde{H}_0(\text{⁻.}) = \mathbb{F}$

We need Tor:

Recall: If M, N are R-modules, then their ~~tensor~~ product $M \underset{R}{\otimes} N$ is

$$\left(\begin{array}{l}\text{free module gen.}\\ \text{by } m\otimes n \quad m\in M,\ n\in N\end{array}\right) \Big/ \left(\begin{array}{l}(m+m')\otimes n = m\otimes n + m'\otimes n\\ m\otimes(n+n') = m\otimes n + m\otimes n'\\ rm\otimes n = m\otimes rn\end{array}\right)$$

Ex.

- $R \underset{R}{\otimes} M \cong M$ because $r\otimes m = 1\otimes rm$
- If M is graded, then
  $R(-a) \underset{R}{\otimes} M \cong M(-a)$
- If $R = \mathbb{F}[x_1 \dots x_n]$,
  $R(-a) \underset{R}{\otimes} \mathbb{F} = \mathbb{F}(-a)$

Given a free resolution of M, and N,

$$0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0 \qquad \mathcal{F}_\circ$$

we get a complex

$$0 \to F_n\otimes N \to \cdots \to F_0\otimes N \to M\otimes N \to 0 \qquad \mathcal{F}_\circ \otimes N$$

with
$$F_i \otimes N \xrightarrow{\partial_i \otimes id} F_{i-1} \otimes N$$
$$f\otimes n \longrightarrow \partial_i f \otimes n$$

This may not be exact, and we let

$$\mathrm{Tor}_i^R(N,M) = i\text{-th homology of } \mathcal{F}_\circ \otimes N.$$

Prop $\mathrm{Tor}_i^R(N,M)$ doesn't depend on the free resolution of M.

Prop $\mathrm{Tor}_i^R(M,N) \cong \mathrm{Tor}_i^R(N,M)$

Pfs Omitted.

Prop If M is ($\mathbb{N}^n$-)graded then
$$\beta_{i,b}(I) = \dim_{\mathbb{F}} \mathrm{Tor}_i^R(\mathbb{F}, M)$$

Pf. $0 \to F_n \to \cdots \to F_0 \to M \to 0 \quad \mathcal{F}_\circ$ minl.

$$\Downarrow$$

$$0 \to F_n\otimes\mathbb{F} \to \cdots \to F_0\otimes\mathbb{F} \to M\otimes\mathbb{F} \to 0$$

- $F_i = \bigoplus_a R(-a)^{\beta_{i,a}} \Rightarrow \boxed{F_i\otimes\mathbb{F} = \bigoplus_a \mathbb{F}(-a)^{\beta_{i,a}}}$
- $\partial_i\otimes id\,(f\otimes a) = \partial_i\otimes id(1\otimes fa) = \partial_i 1\otimes fa = 0\otimes fa = 0$
  $\Rightarrow \boxed{\partial_i\otimes id = 0}$

So $\mathrm{Tor}_i^R(\mathbb{F},M) = F_i\otimes\mathbb{F} = \bigoplus_a \mathbb{F}(-a)^{\beta_{i,a}}$