Pf. Sup. not. Take $f \in I$, $f \notin I'$ of min. degree
Let $f = ax^d + \dots$

- If $d \geq N$:

  Let $a = r_1 a_1 + \dots + r_n a_n$ $\quad (a \in LT(I))$

  $$r_1 f_1 x^{d-e_1} = r_1 a_1 x^d + \dots$$
  $$\vdots$$
  $$\underline{r_n f_n x^{d-e_n}} = \underline{r_n a_n x^d + \dots}$$
  $$I' \ni g = ax^d + \dots$$
  $I' \ni f - g$ ← since $\deg(f-g) < d$

  So $f \in I'$. $\Rightarrow\!\Leftarrow$

- If $d < N$

  Let $a = r_1 b_{d,1} + \dots + r_{n_d} b_{d,n_d}$ $\quad (a \in LT(I_d))$

  $$\underline{r_1 f_{d,1} + \dots + r_{n_d} f_{d,n_d}} = ax^d + \dots$$
  $$I' \ni g$$
  $I' \ni f - g$ ← since $\deg(f-g) < d$

  So $f \in I'$. ∎

So I is finitely generated! ∎

Corollary
- If $\mathbb{F}$ is a field, then $\mathbb{F}[x_1, \dots, x_n]$ is Noetherian
- $\mathbb{Z}[x_1, \dots, x_n]$ is Noetherian

Pf. $\mathbb{F}$, $\mathbb{Z}$ are PIDs ∎

Prop R Noetherian $\Leftrightarrow$ R satisfies the ascending chain condition: If $I_1 \subset I_2 \subset \cdots$ are ideals then for some n we have $I_n = I_{n+1} = I_{n+2} = \cdots$

Pf. See forum ∎

This idea of "leading terms" gives us good generators. How do we do this in $\mathbb{F}[x_1, \dots, x_n]$?

$$LT(2xy^2z - x^3 + 7x^2y^2) = ?$$

- lexicographically: x^3
- degree, then lex: $7x^2y^2$

lex: Fix order, say $x_1 > \dots > x_n$. Say
$$Ax_1^{a_1} x_2^{a_2} \dots x_n^{a_n} \underset{lex}{>} Bx_1^{b_1} x_2^{b_2} \dots x_n^{b_n} \quad ((a_1, \dots, a_n) > (b_1, \dots, b_n))$$
if the first position where they differ has $a_i > b_i$.

grlex Fix order. Say $m_1 \geq m_2$ if
- $\deg m_1 > \deg m_2$, or
- $\deg m_1 = \deg m_2$ and $m_1 \underset{lex}{\geq} m_2$.

Def. A monomial ordering in $\mathbb{F}[x_1, \dots, x_n]$ (or in $\mathbb{Z}^n_{\geq 0}$) is a total order on the set of monomials such that
- $m \geq 1$ for all m
- If $m_1 \geq m_2$ then $mm_1 \geq mm_2$ for all m.

Check:
- lex, grlex are monomial orderings.
- monomial orderings are well orderings (every non-empty set has a minimum element)

Def. Fix a monomial ordering $<$ on $\mathbb{F}[x_1,\dots,x_n]$.

- $f \in \mathbb{F}[x_1,\dots,x_n] \longrightarrow LT(f) = in_<(f) =$ leading term of f
- I ideal in $\mathbb{F}[x_1,\dots,x_n] \longrightarrow LT(I) = in_<(I) = \langle in_<(f) : f \in I \rangle$
- I ideal in $\mathbb{F}[x_1,\dots,x_n] \Rightarrow$ $\{g_1,\dots,g_n\}$ is a Gröbner basis for I if
  - $g_1,\dots,g_n$ generate I
  - $in_<(g_1),\dots,in_<(g_n)$ generate $in_<(I)$

Ex. $I = \langle \underbrace{x^3y - xy^2 + 1}_{f_1}, \underbrace{x^2y^2 - y^3 + 1}_{f_2} \rangle$ $\quad <$: lex with $x > y$.

$in_<(f_1) = x^3y$, $in_<(f_2) = x^2y^2$

But $yf_1 - xf_2 = x - y \in I$, $in_<(x-y) = x$

So $\{f_1, f_2\}$ is not a Gröbner basis. What is?

What do you use a Gröbner basis for?

Computations: Does $f \in I$?

Solve $f_1 = \dots = f_n = 0$.

Find the relations between $f_1,\dots,f_m$

$\vdots$

Theorems: Hilbert's basis theorem

Hilbert's syzygy theorem

$\vdots$

Testing ideal membership: does $f \in I$?

To decide whether $f \in \langle g_1,\dots,g_k \rangle$ we might use:

Division Algorithm

Goal: Write $f = q_1 g_1 + \dots + q_k g_k + r$

- $in(f) \geq in(q_i g_i)$
- r "minimal" (has no monomials divisible by an $in(g_i)$)

Start with $q_1 = \dots = q_k = r = 0$, and then "peel off of f" by succesively cancelling out the "largest" term:

(i) If $in_<(f) = m_i \, in_<(g_i)$, for some i, let

$f \mapsto f - m_i g_i$ (smaller $in_<$)

$q_i \mapsto q_i + m_i$

(ii) If not, let

$f \mapsto f - in_<(f)$ (smaller $in_<$)

$r \mapsto r + in_<(f)$

Repeat until $f \mapsto 0$

In the end we get $f = q_1 g_1 + \dots + q_k g_k + r$, no $in(g_i) \mid in(r)$

Ex: $f = x^2y + y \qquad g_1 = xy + 1 \qquad g_2 = x + y \qquad x > y$

$x^2y + y = \boxed{}(xy+1) + \boxed{}(x+y) + \boxed{}$

$0\,(\underline{xy}+1) + 0\,(\underline{x}+y) + (\underline{x^2y} + y)$

$= x\,(\underline{xy}+1) + 0\,(\underline{x}+y) + (-\underline{x}+y)$

$\boxed{x^2y + y = x\,(xy+1) - 1\,(x+y) + 2y}$

different choices: $\boxed{= 0\,(xy+1) + (xy - y^2)(x+y) + (y^3+y)}$

The answer depends on:

- monomial ordering
- choices in (i)