Theorem. The Hilbert series of the Stanley-Reisner ring R/I_Δ of Δ is

$$H(R/I_\Delta; x) = \frac{\sum_{\sigma\in\Delta}\left[\prod_{i\in\sigma} x_i \prod_{j\notin\sigma}(1-x_j)\right]}{\prod_{i=1}^{n}(1-x_i)}$$

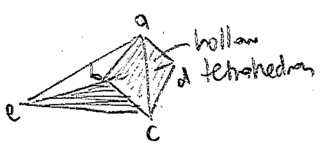
Pf. A basis for R/I_Δ is given by the monomials not in I_Δ

$$\begin{aligned} x_a^{n_a} x_b^{n_b}\cdots x_c^{n_c} \in I_\Delta &\iff \text{some } x_\tau \mid x_a^{n_a}\cdots x_c^{n_c} \quad x_\tau\in I_\Delta \\ (n_i\geq 1) \qquad &\iff x_\tau \mid x_a\cdots x_c \quad \tau\notin\Delta \\ &\iff \{a,\dots,c\}\notin\Delta \end{aligned}$$

So

$$\begin{aligned} H(R/I_\Delta; x) &= \sum_{x^a\notin I_\Delta} x^a \\ &= \sum_{\sigma\in\Delta} \sum_{\substack{x^a\notin I_\Delta \\ \operatorname{supp} a=\sigma}} x^a \\ &= \sum_{\sigma\in\Delta}\prod_{i\in\sigma}\frac{x_i}{1-x_i} \quad \blacksquare \end{aligned}$$

Ex $\Gamma =$



$$\begin{aligned} &1 \\ &+\frac{a}{1-a}+\cdots \\ &+\left(\frac{a}{1-a}\right)\left(\frac{b}{1-b}\right)+\cdots \\ &+\left(\frac{a}{1-a}\right)\left(\frac{b}{1-b}\right)\left(\frac{c}{1-c}\right)+\cdots \end{aligned} = \frac{1-abcd-abe-ace-de+abce+abde+acde}{(1-a)\cdots(1-e)}$$

nice and short! why? we'll see

coarse: $\frac{3x^4-2x^3-x^2+1}{(1-x)^5} = \frac{2x^2+2x+1}{(1-x)^3}$

Def The *f-vector* $(f_{-1}, f_0, f_1, \dots, f_{d-1})$ of Δ is

$f_i := \#$ of i-faces of Δ

The *f-poly* of Δ is

$$\sum_{i=0}^{d} f_{i-1} t^i = f_\Delta(t)$$

The *h-poly* h_Δ and h-vec $(h_0,\dots,h_{d-1})$ are

$$\sum_{i=0}^{d} f_{i-1} t^i (1-t)^{d-i} = (1-t)^d f\left(\frac{t}{1-t}\right) = h_\Delta(t) = \sum h_i t^i$$

Cor With the coarse grading,
$$H(R/I_\Delta; x) = \frac{h_\Delta(x)}{(1-x)^d}$$

Pf Plug in $x, x, \dots, x$ in the fine Hilbert series. $\blacksquare$

Cor $\dim(R/I_\Delta) = 1 + \dim\Delta$

Krull

Pf. $h_\Delta(1) = f_{d-1} > 0$ $\blacksquare$

Two tricks to compute h-vectors: (Stanley / Skandera?)

In ex, $f_\Delta = (1,5,9,5)$

			1	
		1	5	f-vector
	1	4	9	
1	3	5	5	
2	2	0	h-vector	

$$\begin{aligned} f_\Delta(t) &= 1+5t+9t^2+5t^3 \\ &\downarrow \text{invert coeffs} \\ &t^3+5t^2+9t+5 \\ &\downarrow t\mapsto t-1 \\ &(t-1)^3+5(t-1)^2+9(t-1)+5 \\ &= t^3+2t^2+2t\ (+0) \\ &\downarrow \text{invert order} \\ h_\Delta(t) &= 1+2t+2t^2 \end{aligned}$$

Dehn-Somerville Relations. If a simplicial complex Δ is the boundary of a d-polytope, h_Δ is symmetric.

It turns out that free resolutions of squarefree monomial ideals are very closely related to homology groups of simplicial complexes. So let's learn that.

Algebraic topology:

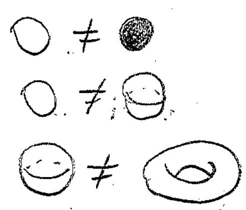
Top space $X \longrightarrow$ Alg. object $A(X)$.

so that if X "=" Y then $A(X)$ "=" $A(Y)$

homeomorphic
homotopic

Exs. square "=" circle

coffee cup "=" donut (homeomorphic)

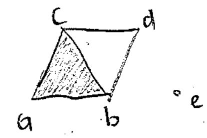


By triangulating a surface, we make it a simplicial complex.

How do we detect the topology of a simplicial complex algebraically?

E.g., how do we "find holes"?

Intuition: holes are cycles which aren't boundaries.



$= \Delta$ $\qquad \mathbb{F}$

$F_i(\Delta) = \{i\text{-faces of } \Delta\}$

$C_i(\Delta) = \mathbb{F}^{F_i(\Delta)}$ = vector space with basis $\{e_\sigma\}_{\sigma \in F_i(\Delta)}$

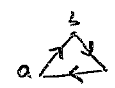
In ex,

$C_{-1}(\Delta) = \mathbb{F}^{\{\emptyset\}} = \{\alpha \cdot e_\emptyset : \alpha \in \mathbb{F}\}$

$C_0(\Delta) = \mathbb{F}^{\{a,b,c,d,e\}} = \{\alpha e_a + \beta e_b + \gamma e_c + \delta e_d + \varepsilon e_e : \alpha, \dots, \varepsilon \in \mathbb{F}\}$

$C_1(\Delta) = \mathbb{F}^{\{ab, ac, bc, bd, cd\}} = \dots$

$C_2(\Delta) = \mathbb{F}^{\{abc\}} = \dots$

The boundary of  is 

$$\partial_i : C_i(\Delta) \longrightarrow C_{i-1}(\Delta)$$

$$e_{\{a_1 \cdots a_{i+1}\}} \longmapsto \sum_{j=1}^{i+1} (-1)^j e_{\{a_1 \cdots \hat{a}_j \cdots a_{i+1}\}}$$

This is the i-th boundary map.

$\boxed{\text{Prop. } \partial_{i-1} \circ \partial_i = 0}$

Pf. $\partial_{i-1} \circ \partial_i (e_{\{a_1 \cdots a_{i+1}\}}) = \partial_{i-1}\left(\sum_{j=1}^{i+1} (-1)^j e_{\{a_1 \cdots \hat{a}_j \cdots a_{i+1}\}}\right)$

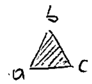
$$= \sum_{j=1}^{i+1}\left[(-1)^j\left(\sum_{k<j} (-1)^k e_{\{a_1 \cdots \hat{a}_k \cdots \hat{a}_j \cdots a_{i+1}\}} + \sum_{k>j} (-1)^{k-1} e_{\{a_1 \cdots \hat{a}_j \cdots \hat{a}_k \cdots a_{i+1}\}}\right)\right]$$

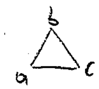
$$= \sum_{r<s} e_{\{a_1 \cdots \hat{a}_r \cdots \hat{a}_s \cdots a_{i+1}\}}\left[(-1)^s(-1)^r + (-1)^r(-1)^{s-1}\right] = 0 \quad \blacksquare$$

So we have the <u>(augmented/reduced) chain cx</u> of Δ:

$$0 \to C_{d-1}(\Delta) \xrightarrow{\partial_{d-1}} \cdots \xrightarrow{\partial_1} C_0(\Delta) \xrightarrow{\partial_0} C_{-1}(\Delta) \to 0$$

Let

$B_i(\Delta) = \operatorname{Im} \partial_{i+1}$ "boundaries" 

$Z_i(\Delta) = \ker \partial_i$ "cycles" 

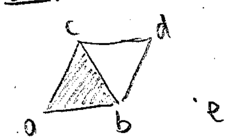
and

$\widetilde{H}_i(\Delta) = \ker \partial_i / \operatorname{Im} \partial_{i+1}$

think: i-dim holes (cycles which are not boundaries)

is the i-th <u>reduced homology group</u> of Δ.

Ex.



$= \Delta$

$C_{-1}(\Delta) = \operatorname{span}(e_\emptyset)$
$C_0(\Delta) = \operatorname{span}(e_a, e_b, e_c, e_d, e_e)$
$C_1(\Delta) = \operatorname{span}(e_{ab}, e_{ac}, e_{bc}, e_{bd}, e_{cd})$
$C_2(\Delta) = \operatorname{span}(e_{abc})$

$\widetilde{H}_1(\Delta) = \mathbb{F}$

$\operatorname{span}(e_{ab} - e_{ac} + e_{bc},\ e_{bc} - e_{bd} + e_{cd}) / \operatorname{span}(e_{ab} - e_{ac} + e_{bc})$

basis: $e_{bc} - e_{bd} + e_{cd}$

$\widetilde{H}_0(\Delta) = \mathbb{F}$

$\operatorname{span}(\text{all } e_i - e_j) / \operatorname{span}(e_a - e_b, e_a - e_c, e_b - e_c, e_b - e_d, e_c - e_d)$

basis: $e_d - e_e$

$\widetilde{H}_{-1}(\Delta) = 0$
$\operatorname{span}(e_\emptyset) / \operatorname{span}(e_\emptyset) = 0$

$$0 \to \underset{C_2}{\mathbb{F}} \xrightarrow[\partial_2]{\begin{array}{c|c} & abc \\ \hline ab & -1 \\ ac & 1 \\ bc & -1 \\ bd & 0 \\ cd & 0 \end{array}} \underset{C_1}{\mathbb{F}^5} \xrightarrow[\partial_1]{\begin{array}{c|ccccc} & ab & ac & bc & bd & cd \\ \hline a & 1 & 1 & 0 & 0 & 0 \\ b & -1 & 0 & 1 & 1 & 0 \\ c & 0 & -1 & -1 & 0 & 1 \\ d & 0 & 0 & 0 & -1 & -1 \\ e & 0 & 0 & 0 & 0 & 0 \end{array}} \underset{C_0}{\mathbb{F}^5} \xrightarrow[\partial_0]{\begin{array}{c|ccccc} & a & b & c & d & e \\ \hline \emptyset & -1 & -1 & -1 & -1 & -1 \end{array}} \underset{C_{-1}}{\mathbb{F}} \to 0$$

<u>Ex</u> <u>The boundary of the (n-1)-simplex</u> (The sphere S^{n-2})

$\Delta = \partial \Delta^{n-1} = \{ E \subset [n] \mid E \neq [n] \}$

n=3 

$C_{i-1}(\Delta) = \operatorname{span}(e_S)_{\substack{S \subset [n] \\ |S| = i}} \cong \mathbb{F}^{\binom{n}{i}}$

The complex is

$$0 \to \mathbb{F}^{\binom{n}{n-1}} \xrightarrow[\partial_{n-2}]{} \mathbb{F}^{\binom{n}{n-2}} \to \cdots \xrightarrow[\partial_1]{} \mathbb{F}^{\binom{n}{1}} \xrightarrow[\partial_0]{} \mathbb{F}^{\binom{n}{0}} \to 0 \qquad \mathcal{F}_\bullet$$

which is almost like the free resol. of $k[x_1 \ldots x_n] / \langle x_1 \ldots x_n \rangle$ in HW2. That one gave an exact sequence:

$$0 \to R^{\binom{n}{n}} \to R^{\binom{n}{n-1}} \to \cdots \to R^{\binom{n}{1}} \to R^{\binom{n}{0}} \to 0$$

with basically the same maps.

In the same way, $\mathcal{F}_\bullet$ is exact except at $\mathbb{F}^{\binom{n}{n-1}} = C_{n-2}(\Delta)$, where

$\widetilde{H}_{n-2}(\Delta) = \ker \partial_{n-2} / \operatorname{Im} 0 = \operatorname{Im} \text{"}\partial_{n-1}\text{"}$
$= \operatorname{span}(e_{23\ldots n} - e_{13\ldots n} + \cdots \pm e_{12\ldots n-1})$

So $\widetilde{H}_i(S_{n-2}) = \begin{cases} \mathbb{F} & i = n-2 \\ 0 & \text{otherwise} \end{cases}$

Two topological spaces X and Y are <u>homeomorphic</u> if there is $f: X \to Y$ such that

- f is bijective
- f is continuous
- f^{-1} is continuous

Think: deform X to Y continuously.

Ex:

 , 	yes	
$(0,1)$, $\mathbb{R}$	yes	
donut, mug	yes	
$\mathbb{R}^n$, $\mathbb{R}^m$	no	$(m \neq n)$
$\bullet$, n-simplex	no	$(n \geq 1)$

> **Thm** If X and Y are homeomorphic, they have the same homology groups.

Two maps $f_0, f_1 : U \to V$ are <u>homotopic</u> if there is a continuous $f: U \times [0,1] \to V$ with $f(u,0) = f_0(u)$, $f(u,1) = f_1(u)$

Ex: f_0 :  $\to$  $f_0(u) = \bullet$

f_1 :  $\to$  $f_1(u) = u$

$f_t(u) =$ homothety by t

X and Y are <u>homotopy equivalent</u> if there are $f: X \to Y$ and $g: Y \to X$ such that

- $f \circ g$ is homotopic to id_Y
- $g \circ f$ is homotopic to id_X

> **Thm** If X and Y are homotopy equivalent then they have the same homology groups.

Ex $X = B^n = \{x \in \mathbb{R}^n \mid \sum_{i=1}^n x_i^2 \leq 1\}$

$Y = \{0\} \subset \mathbb{R}^n$

Let $f: X \to Y$, $x \mapsto 0$ $\qquad g: Y \to X$, $0 \mapsto 0$

Then $f \circ g : Y \to Y$, $0 \mapsto 0$ is the identity $\qquad g \circ f : X \to X$, $x \mapsto 0$ is homotopic to the identity

Since $\begin{cases} B^n \text{ and } \bullet \text{ are homotopic} \\ B^n \text{ and } \underset{\substack{\uparrow \\ \text{simplex}}}{\Delta^n} \text{ are homeomorphic} \end{cases}$

Then $\tilde{H}_i(\Delta^n) = \tilde{H}_i(\bullet) = 0$ for all i.

So the chain complex for Δ^{n-1} is exact:

$$0 \to \mathbb{F}^{\binom{n}{n}} \xrightarrow{\partial_{n-1}} \mathbb{F}^{\binom{n}{n-1}} \xrightarrow{\partial_{n-2}} \cdots \to \mathbb{F}^{\binom{n}{1}} \xrightarrow{\partial_0} \mathbb{F}^{\binom{n}{0}} \to 0$$

Hence the chain complex for $\partial \Delta^{n-1} = S^{n-2}$:

$$0 \to \mathbb{F}^{\binom{n}{n-1}} \xrightarrow{\partial_{n-2}} \cdots \to \mathbb{F}^{\binom{n}{1}} \xrightarrow{\partial_0} \mathbb{F}^{\binom{n}{0}} \to 0$$

is exact except at C_{n-2} where $\tilde{H}_{n-2}(\partial \Delta^{n-1}) = \mathbb{F}$.

This gives another proof for the homology of the sphere.