Ok, so let us focus (for a while!) on

MONOMIAL IDEALS

(following Miller-Sturmfels)

A <u>monomial</u> <u>ideal</u> $I \subset \mathbb{F}[\mathbf{x}] = \mathbb{F}[x_1, \dots, x_n]$ is one generated by monomials:

$$I = \langle \mathbf{x}^{a_1}, \dots, \mathbf{x}^{a_n} \rangle$$

There is a unique minl. generating set. It is finite.

SQUARE-FREE MONOMIAL IDEALS (Stanley-Reisner)

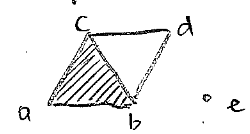
$\mathbf{x}^a$ is square-free if each $a_i \in \{0,1\}$

I is <u>squarefree</u> if its generators are

<u>Ex</u>. $I = \langle ad, ae, bcd, be, ce, de \rangle$ in $\mathbb{F}[a,b,c,d,e]$

Let $\Delta_I = \{$supports of squarefree monomials not in $I\}$

Here $\Delta_I = \{1, a, b, c, d, e, ab, ac, bc, bd, cd, abc\}$ are the faces of the simplicial complex



<u>Def</u> An <u>abstract simplicial complex</u> on E is a collection $\mathcal{F}$ of subsets of E "faces/simplices" with $\left(\begin{matrix} G \in \mathcal{F} \\ F \subset G \end{matrix}\right) \Rightarrow F \subset \mathcal{F}$ Say $\dim F = |F| - 1$.

Think: points, lines, triangles, tetrahedra, ..., simplices
glued along their common face

Theorem

There is a bijection

$$\begin{pmatrix}\text{simplicial complexes} \\ \text{on } [n]\end{pmatrix} \leftrightarrow \begin{pmatrix}\text{squarefree monomial} \\ \text{ideals in } \mathbb{F}[x_1 \dots x_n]\end{pmatrix}$$

Pf.

Given Δ a simplicial complex, let

$$I_\Delta = \langle x_{i_1} \cdots x_{i_k} \mid \{i_1, \dots, i_k\} \notin \Delta \rangle$$

Given an ideal I, let

$$\Delta(I) = \{\{i_1, \dots, i_k\} \mid x_{i_1} \cdots x_{i_k} \notin I\} \quad \text{(a s.c.)}$$

These are clearly inverse of each other ■

Def I_Δ = Stanley-Reisner ideal of Δ

R/I_Δ = Stanley-Reisner ring of Δ

Let $m^\tau = \langle x_i \mid i \in \tau \rangle$ for $\tau \subset [n]$. (prime ideal)

Prop $I_\Delta = \bigcap_{\sigma \in \Delta} m^{[n]-\sigma}$

Pf. $\subseteq$: Let $x_\tau = \prod_{i \in \tau} x_i$ be a gen. of LHS, $\tau \notin \Delta$.

Sup $x_\tau \notin m^{[n]-\sigma} = \langle x_{a_1}, \dots, x_{a_k} \rangle$ $\sigma \in \Delta$

Then $a_1 \notin \tau, \dots, a_k \notin \tau$, so $[n]-\sigma \subset [n]-\tau$

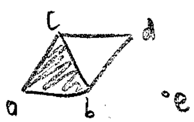
so $\tau \subseteq \sigma$ $\Rightarrow\Leftarrow$

$\supseteq$: RHS is monomial.

A monomial $x_{i_1}^{a_1} \cdots x_{i_k}^{a_k}$ is in RHS iff

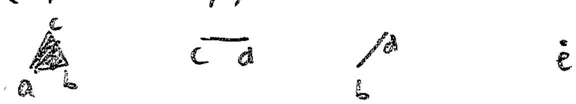
iff $\{i_1, \dots, i_k\}$ contains an elt of all $[n]-\sigma$

iff $\{i_1, \dots, i_k\} \not\subset \sigma$ for all $\sigma \in \Delta$. ■

Ex 

$I_\Delta = \langle ad, ae, bcd, be, ce, de \rangle$ (min'l non-faces)

$= \langle d, e \rangle \cap \langle a, b, e \rangle \cap \langle a, c, e \rangle \cap \langle a, b, c, d \rangle$



(max'l faces)

Note If $\mathbb{F}$ is infinite, the following are in bijection:

- simplicial complexes on $[n]$
- squarefree monomial ideals in $\mathbb{F}[x_1 \dots x_n]$
- unions of coordinate subspaces in $\mathbb{F}^n$

(In ex, subspaces are $\underset{d=e=0}{abc}$, cd, bd, e.)