Corollary (Hilbert) A graded R-module M has

$$H(M;x) = \frac{K(M;x)}{(1-x)^n} \quad K \text{ polynomial}.$$

$H_M(d)$ equals a polynomial in d of $\deg \le n$ for large enough d.

$M = R = \mathbb{F}$-algebra
n = Krull dimension
= max length of $I_1 \subset \cdots \subset I_k$ prime ideals
= $\text{transc.}\deg_{\mathbb{F}} R$

In ex, $1 = \frac{1}{(1-x)^3} - 3\frac{x}{(1-x)^3} + 3\frac{x^2}{(1-x)^3} - \frac{x^3}{(1-x)^3}$

In practice, this works better:

Theorem (Macaulay)

Let P be a fin. gen graded R-module, given as $P = F/M$. (F = free module with homog. basis, M = submodule gen by homog. elts (relations))

Then

F/M and $F/\text{in}M$ have the same Hilbert function.

Pf Let B = {monomials in F, not in $\text{in}(M)$}

Claim: $\overline{B}$ is a basis for $P = F/M$
$\overline{B}_d$ is a basis for P_d

Pf lin-ind: Sup $\sum_i \lambda_i \overline{m_i} = \overline{0}$ in P $\quad \lambda_i \in \mathbb{F}, m_i \in B$

$\sum_i \lambda_i m_i = m \in M$

so $\text{in}(m)$ is one of the m_i

$\Rightarrow m_i \notin B$.

Span. Suppose $\overline{f} \in F/M$ is not gen by $\overline{B}$

Then $f \in F$ is not gen by $M \cup B$.

Take such an f with $\text{in}(f)$ minimal

- If $\text{in}(f) \in B$, then $g = f - \text{in}(f)$ is not gen by $M \cup B$ has $\text{in}(g) < \text{in}(f)$.
- If $\text{in}(f) \notin B$ then $\text{in}(f) \in \text{in}M$ so take

$m \in M \quad m = \text{in}(f) + m'$ (larger)
$f = \text{in}(f) + f'$

$$\underbrace{f - m}_{\text{not gen. by } M\cup B} = \underbrace{f' - m'}_{\text{smaller initial term}}$$

$\blacksquare$

Note $F/\text{in}M \cong \bigoplus R/I_i$, so

Computing Hilb(arbitrary module) is reduced to Computing Hilb(R/I) for monomial ideals I

This is NP-hard (Bayer–Stillman), but can be done reasonably for small/"nice" I.

A recursive procedure for $H_{M/I}(n)$:

Let $I = \langle m_1, \dots, m_k \rangle$

Define: $n = m_k$, $d = \deg n$

$I' = \langle m_1, \dots, m_{k-1} \rangle$

$J = (I' : m_k) = \{ m \mid m \cdot n \in I' \}$

$= \left\langle \frac{m_1}{\gcd(m_1, n)}, \dots, \frac{m_{k-1}}{\gcd(m_{k-1}, n)} \right\rangle$

Then there is an exact sequence of R-modules

$$0 \to (R/J)(-d) \xrightarrow{f} R/I' \xrightarrow{g} R/I \to 0$$

so

$$H_{R/I}(m) = H_{R/I'}(m) - H_{R/J}(m-d)$$

(I', J have fewer gens.)

(HW Exercise.)

There are other procedures.

ideals

(By the way:
If I is a homogeneous ideal in R,

$$H(R/I; x) = H(R; x) - H(I; x)$$

$$H(R/I; x) = \frac{1}{(1-x)^n} - H(I; x)$$

So computing the Hilbert series of I, R/I are equivalent questions.)

This allowed Macaulay to completely classify the Hilbert functions of graded rings (with some conditions)

Fact

For fixed n, k there is a unique expression

$$n = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \dots + \binom{a_i}{i} \qquad a_k > a_{k-1} > \dots > a_i \geq i \geq 1$$

Let

$$\partial^k(n) = \binom{a_k - 1}{k-1} + \binom{a_{k-1} - 1}{k-2} + \dots + \binom{a_i - 1}{i-1}$$

Theorem (Macaulay)
For $(f_0, f_1, \dots) \in \mathbb{N}^\infty$ the following are equivalent:

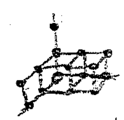
(i) $f_0 = 1$ and $\partial^k(f_k) \leq f_{k-1}$ for $k \geq 2$

(ii) There exists a graded ring R, with $R_0 \cong \mathbb{F}$ (field) and R_1 generating R, so that $H_R(i) = f_i$

(iii) There is a multicomplex with f-vector f.

(A multicomplex M is a set of (monic) monomials such that $m \in M$, $n \mid m \Rightarrow n \in M$. Its f-vector is $(f_0, f_1, \dots)$ where $f_i = $ # of monomials of deg i)

Ex: $(1, 3, 6, 4, 1)$



(Many nice variants of this result.)