Lemma. Suppose $\{g_1, \ldots, g_n\}$ are a G.b. in R^m, arranged so that:

When $\begin{pmatrix} in(g_i) = n_i e_v \\ in(g_j) = n_j e_v \end{pmatrix}$, $i<j \Leftrightarrow n_i > n_j$ in lex order

If $x_1, \ldots, x_s$ are missing among the $in(g_i)$, then $x_1, \ldots, x_s, x_{s+1}$ are missing among the $in(\tau_{ij})$

Pf. $in(\tau_{ij}) = \begin{cases} 0 & \text{if } in(g_i), in(g_j) \text{ involve different } e\text{'s} \\ m_{ji} e_v & \text{if } in(g_i) = n_i e_v \quad in(g_j) = n_j e_v, \end{cases}$

where $m_{ij} = \dfrac{\operatorname{lcm}(n_i, n_j)}{n_i}$.

This involves no $x_1, \ldots, x_s$, and n_i has more x_{s+1} than n_j, so in x_{s+1} this is $\dfrac{x_{s+1}^a}{x_{s+1}^a}$. ∎

Pf of Hilbert's syzygy thm

Arranging gens of syzygy modules, I can get

$Syz^1(M) \longrightarrow$ in (gens) involve ?

$Syz^2(M) \longrightarrow$ in (gens) involve no x_1

$\vdots$

$Syz^n(M) \longrightarrow$ in (gens) involve no $x_1, x_2, \ldots, x_{n-1}$

$Syz^{n+1}(M) \longrightarrow$ in (gens) - involve no variables
- can't be constants $\neq 0$

$\Rightarrow Syz^{n+1}(M) = 0.$ ∎

Hilbert functions and series

(Measuring rings, ideals, modules, ...)

A ring R is graded if it can be decomposed as

$$R = R_0 \oplus R_1 \oplus R_2 \oplus \cdots \quad \text{as abelian groups}$$

so that $R_i R_j \subseteq R_{i+j}$

It is $\mathbb{N}^n$-graded if

$$R = \bigoplus_{a \in \mathbb{N}^n} R_a$$

so that $R_a R_b = R_{a+b}$

Ex $R = \mathbb{F}[x_1, \ldots, x_n]$

graded: $R_i = \{$homog. polys of deg i in $R\}$

$\mathbb{N}^n$-graded: $R_a = \{$homog. polys of deg a in $R\}$

↳ monom. $m\mathbf{x}^a = m x_1^{a_1} \cdots x_n^{a_n}$

A graded module M over a graded ring R is

$$M = \bigoplus_{i \geq 0} M_i \quad \text{so that } R_i M_j \subseteq M_{i+j}$$

and an $\mathbb{N}^n$-graded module over an $\mathbb{N}^n$-graded ring R is

$$M = \bigoplus_{a \in \mathbb{N}^n} M_a \quad \text{so that } R_a M_b \subseteq M_{a+b}$$

Ex $M = R(-a) \cong \langle x^a \rangle$ free module gen. in degree a

The <u>Hilbert fn/series</u> of a graded module

$$M = \bigoplus_{i \geq 0} M_i \qquad\qquad M = \bigoplus_{a \in \mathbb{N}^n} M_a$$

$$H_M(i) = \dim_{\mathbb{F}} M_i \qquad\qquad H_M(a) = \dim_{\mathbb{F}} M_a$$

$$H(M;x) = \sum_{i \geq 0} (\dim_{\mathbb{F}} M_i) x^i \qquad\qquad H(M;x) = \sum_{a \in \mathbb{N}^n} (\dim_{\mathbb{F}} M_a) x^a$$

coarse $\qquad$ fine

$$H(M;x) = H(M;x,x,\dots,x)$$

<u>Ex.</u>

- $M = R = \mathbb{F}[x_1,\dots,x_n]$
- $H_R(i) = \left(\!\binom{n}{i}\!\right) = \binom{n+i-1}{i-1}$

$$H(R;x) = \sum_{i \geq 0} \left(\!\binom{n}{i}\!\right) x^i = \sum_{i \geq 0} (-1)^i \binom{-n}{i} x^i = \frac{1}{(1-x)^n}$$

- $H_R(a) = 1$

$$H(R;x) = \sum_{a \in \mathbb{N}^n} x^a = \frac{1}{(1-x_1)\cdots(1-x_n)}$$

- $M = R(-b)$

$$H_M(a) = \begin{cases} 1 & a \geq b \text{ componentwise} \\ 0 & \text{ow} \end{cases}$$

$$H(M;x) = \frac{x^a}{\prod_{i=1}^n (1-x_i)}$$

- $M = \bigoplus_{i=1}^r R(-b_i)$

$$H(M;x) = \frac{\sum_{i=1}^r x^{b_i}}{\prod_{i=1}^n (1-x_i)}$$

A <u>graded free resolution</u> is one where the R-modules are graded, and the maps preserve degree

<u>Ex.</u>

$$0 \to R(-3) \xrightarrow{\begin{bmatrix} z \\ -y \\ x \end{bmatrix}} R(-2)^3 \xrightarrow{\begin{bmatrix} y & z & 0 \\ -x & 0 & z \\ 0 & -x & -y \end{bmatrix}} R(-1)^3 \xrightarrow{[x\ y\ z]} R(0) \xrightarrow{\epsilon} M \to 0$$

<u>Prop</u> If a graded R-module has a finite graded free resolution

$$\mathcal{F}: 0 \to F_r \to \cdots \to F_0 \to M \to 0$$

then

$$H_M(i) = H_{F_0}(i) - H_{F_1}(i) + \cdots + (-1)^r H_{F_r}(i)$$

$$H(M;x) = H(F_0;x) - H(F_1;x) + \cdots + (-1)^r H(F_r;x)$$

<u>Proof.</u> In degree i, get an exact sequence of vector spaces

$$0 \to (F_r)_i \to \cdots \to (F_0)_i \to M_i \to 0$$

of dims $H_{F_r}(i)$, $\dots$, $H_{F_0}(i)$, $H_M(i)$

For such a seq

$$0 \to V_r \xrightarrow{\partial_r} V_{r-1} \xrightarrow{\partial_{r-1}} \cdots \xrightarrow{\partial_1} V_0 \xrightarrow{\partial_0} V \to 0$$

we have

$$\xrightarrow{\partial_i} V_i \xrightarrow{\partial_{i-1}} V_{i-1}$$

$$\operatorname{im} \partial_{i-1} \cong V_i / \ker \partial_{i-1} = V_i / \operatorname{im} \partial_i$$

so

$$\dim(\operatorname{im} \partial_{i-1}) = \dim V_i - \dim(\operatorname{im} \partial_i)$$

so

$$\dim V = \dim V_0 - \dim(\operatorname{im} \partial_0)$$
$$= \dim V_0 - (\dim V_1 - \dim \operatorname{im} \partial_1) = \cdots$$