Combinatorial commutative algebra

sfsu: math 850
los andes: mate 3117

- combinatorial techniques to study commutative rings
- algebraic techniques to study combinatorial objects

(topology, discrete geometry appear naturally.)

Some topics:

- Gröbner bases
- Stanley-Reisner rings and simplicial complexes
- Monomial ideals
- Semigroup rings and polytopes

① Gröbner Bases (Dummit + Foote, Sec. 9.6.)

Def. A comm. ring R with 1 is Noetherian if every ideal of R is finitely generated.

(Compare: PID $\Leftrightarrow$ every ideal is ppal)

Ring: $(R, +, \cdot)$
- $(R,+)$ comm. group
- $\cdot$ assoc.
- $+, \cdot$ distrib

Ideal I: Subring I with $IR \subseteq R$ $RI \subseteq R$
$(a, b \in I, r \in R \Rightarrow ar - b \in I)$

Hilbert's Basis Theorem:
R Noetherian $\Rightarrow R[x]$ Noetherian

Pf Sup. I ideal in $R[x]$

- Let $LC(I) = \{\text{leading coeffs. of elts of } I\}$
  Check: $LC(I)$ ideal in R.
  So let $LC(I) = \langle a_1, \ldots, a_n \rangle$ (R Noetherian)
  $f_i = a_i x^{e_i} + \ldots$
  $N = \max e_i$
- Let $LC(I_d) = \{\text{leading coeffs of elts of } I \text{ of deg } d\} \cup \{0\}$
  Check: $LC(I_d)$ ideal in R
  So $LC(I_d) = \langle b_{d,1}, \ldots, b_{d,n_d} \rangle$
  $f_{d,i} = b_{d,i} x^d + \ldots$

Claim. $I = \langle \underset{1 \le i \le n}{f_i}, \underset{\substack{1 \le d < N \\ 1 \le i \le n_d}}{f_{d,i}} \rangle =: I'$