4 If $\triangle_1$ and $\triangle_2$ are simplicial complexes on disjoint sets E_1 and E_2, we define the *join* $\triangle_1 * \triangle_2$ to be the simplicial complex on $E_1 \cup E_2$ whose faces are the sets $A_1 \cup A_2$ with $A_1 \in \triangle_1$ and $A_2 \in \triangle_2$. Compute the h-vector of $\triangle_1 * \triangle_2$ in terms of the h-vectors of $\triangle_1$ and $\triangle_2$.

Let $\triangle_1$ and $\triangle_2$ be simplicial complexes on disjoint sets E_1 and E_2 with f-vectors $(a_{-1}, a_0, \ldots, a_{n-1})$ and $(b_{-1}, b_0, \ldots, b_{m-1})$ respectively.

CLAIM: $f_{\triangle_1 * \triangle_2}(x) = f_{\triangle_1}(x) \cdot f_{\triangle_2}(x)$

Since $f_{\triangle_1}(x) \cdot f_{\triangle_2}(x) = \sum_{j=0}^{n} a_{j-1}x^j \cdot \sum_{k=0}^{m} b_{k-1}x^k = \sum_{j=0}^{n+m} (\sum_{k=0}^{j} a_{k-1}b_{j-k-1})x^j$ (by the definition of multiplication of polynomials), to prove the above claim, we need to show that the number of i-faces of $\triangle_1 * \triangle_2$ is equal to $\sum_{k=0}^{i+1} a_{k-1}b_{i-k}$.

Let $-1 \leq i \leq n+m-1^{\dagger}$. Since E_1 and E_2 are disjoint, for all $A_1 \in \triangle_1$ and $A_2 \in \triangle_2$, $|A_1 \cup A_2| = |A_1| + |A_2|$. So every i-face of $\triangle_1 * \triangle_2$ is of the form $A_1 \cup A_2$ such that $A_1 \in \triangle_1$, $A_2 \in \triangle_2$, and $|A_1| + |A_2| = i+1$. This means that if $A_1 \cup A_2$ is an i-face of $\triangle_1 * \triangle_2$, then $|A_1| = k$ iff $|A_2| = i+1-k$. Furthermore, since there are a_{k-1} elements of

$\triangle_1$ with cardinality k and b_{i-k} elements of $\triangle_2$ with cardinality $i+1-k$ (this is the definition of the f-vectors), then the number of ways to choose an element, A_1, from $\triangle_1$, and an element, A_2, from $\triangle_2$, such that $|A_1| + |A_2| = i+1$ is preceisely $\sum_{k=0}^{i+1} a_{k-1} b_{i-k}$. Therefore, $f_{\triangle_1 * \triangle_2}(x) = f_{\triangle_1}(x) \cdot f_{\triangle_2}(x)$.

Now let us use this identity to compute $h_{\triangle_1 * \triangle_2}(x)$.

$$
\begin{aligned}
h_{\triangle_1 * \triangle_2}(x) &= (1-x)^{n+m} f_{\triangle_1 * \triangle_2}\left(\frac{x}{1-x}\right) \\
&= (1-x)^n (1-x)^m f_{\triangle_1}\left(\frac{x}{1-x}\right) \cdot f_{\triangle_2}\left(\frac{x}{1-x}\right) \\
&= (1-x)^n f_{\triangle_1}\left(\frac{x}{1-x}\right) \cdot (1-x)^m f_{\triangle_2}\left(\frac{x}{1-x}\right) \\
&= h_{\triangle_1}(x) \cdot h_{\triangle_2}(x)
\end{aligned}
$$

So if the h vectors for $\triangle_1$ and $\triangle_2$ are $(c_{-1}, c_0, \ldots c_{n-1})$ and $(d_{-1}, d_0, \ldots d_{m-1})$, respectively, then the h vector of $\triangle_1 * \triangle_2$ is $(h_{-1}, h_0, \ldots h_{n+m-1})$ where $h_i = \sum_{k=0}^{i+1} c_{k-1} d_{i-k}$.