5 Compute the Hilbert function and series of the ring $\mathbb{C}[w,x,y,z]/\langle wy - z^2, wx - yz, xz - y^2\rangle$.

Let $R = \mathbb{C}[w,x,y,z]$ and $I = \langle wy - z^2, wx - yz, xz - y^2\rangle$. By Macaulay's Thm, the Hilbert function and series for R/I is equal to the Hilbert function and series for $R/(\text{in } I)$.

Since $\{wy - z^2, wx - yz, xz - y^2\}$ is a Grobner basis for I (verified by Mathematica), in $I = \langle wy, wx, xz\rangle$. This means that every element of $R/(\text{in } I)$ is of the form $a + I$ where a is a polynomial with any constant in $\mathbb{C}$ and monomial terms in $\mathbb{C}[y,z] \cup \mathbb{C}[w,z] \cup \mathbb{C}[x,y]$.

So, for each $d \geq 1$, the number of basis elements of $(R/\text{in } I)_d$ is equal to the number monomials in $\mathbb{C}[y,z]$ with degree d, $\binom{d+1}{1}$, plus the number of monomials in $\mathbb{C}[w,z]$ with degree d minus the number of these monomials containing only z, $\binom{d+1}{1} - 1$, plus the number of monomials in $\mathbb{C}[x,y]$ with degree d minus the number of these monomials containing only y, $\binom{d+1}{1} - 1$. Thus, computing $H_{R/I}(d) = H_{R/(\text{in } I)}(d)$, we get:

$$H_{R/I}(d) = H_{R/(\text{in } I)}(d) = \begin{cases} 1 & \text{d=0 (1 is the only basis element when d=0)} \\ 3(d+1)-2 & \text{otherwise.} \end{cases}$$

And

$$\begin{aligned} H(R/I;x) = H(R/(\text{in } I);x) &= 1 + \sum_{d\geq 1}(3(d+1)-2)x^d \\ &= 1 + 3\cdot\sum_{d\geq 1}(d+1)x^d - 2\cdot\sum_{d\geq 1}x^d \\ &= 1 + 3\cdot\frac{d}{dx}(\sum_{d\geq 1}x^{d+1}) - 2\cdot\sum_{d\geq 1}x^d \\ &= 1 + 3\cdot\frac{2x-x^2}{(1-x)^2} - 2\cdot\frac{x}{1-x} \\ &= \frac{1+2x}{(1-x)^2} \end{aligned}$$

Credits: I worked Zoe, Fang-i, Andrew, and Lothar