4. **Hilbert series of monomial ideals**. Compute the Hilbert function and series for the ring $[w,x,y,z] \big/ \langle \langle w,x \rangle \cap \langle y,z \rangle \rangle$

Let $I = \langle w,x \rangle$ and $J = \langle y,z \rangle$. Then $I \cap J = (tI + (1-t)J) \cap \ [w,x,y,z]$ from proposition 30 pg 330 Dummit and Foote.

$(tI + (1-t)J) = \langle tw, tx, y - ty, z - tz \rangle$ and a Grober basis G for $(tI + (1-t)J)$ is given by $G = \{xz, xy, wz, wy, -z + tz, -y + ty, tx, tw\}$. (from Mathematica)
$\Rightarrow \ (tI + (1-t)J) \cap \ [w,x,y,z] = \langle xz, xy, wy, wz \rangle = I \cap J$.

Let $M = \ [w,x,y,z] \big/ \langle wy, wz, xy, xz \rangle$ and $H_M(i) = \dim \ (M_i)$ denote the Hilbert function.

Since $\{w,x,y,z\}$ forms a basis for the vector space M_1 we have $H_M(1) = 4$.
$\{w^2, x^2, y^2, z^2, wx, yz\}$ forms a basis for M_2 so $H_M(2) = 6$.
For M_3 we have the usual basis elements in one letter of degree three, w^3, x^3, y^3, z^3, and four basis elements in two letters of degree three w^2x, wx^2, y^2z, yz^2. So $H_M(3) = 8$.

Notice that any monomial in three or four letters is already a multiple of one of the generators, wy, wz, xy, xz. These monomials are never basis elements of M_i for any $i \in$, so they needn't be counted.

To calculate $H_M(d)$ we go through the following process.

Count the four monomials in one letter of degree d, w^d, x^d, y^d, z^d.
Now count the basis elements in two letters. These are always multiples of wx, yz.
Since there are only two possible letters the number of monomials in two letters are $2\left\langle {d-2 \atop 2} \right\rangle = 2\binom{d-1}{d-2} = 2(d-1)$. So my formula for the Hilbert function becomes $H_M(d) = 2(d-1) + 4 = 2d + 2$, and note that $H_M(0) = 1$. The Hilbert series is given by

$$H(M;x) = \sum_{i=0}^{\infty} H_M(i)x^i = 1 + \sum_{i=1}^{\infty} (2i+2)x^i$$
$$= 1 + 2[x\sum_{i=1}^{\infty} ix^{i-1} + \sum_{i=1}^{\infty} x^i]$$

$$= 1 + 2[\sum_{i-1}^{\infty} (x^i)' + \frac{x}{1-x}]$$

$$= 1 + 2[x(\frac{x}{1-x})' + \frac{x}{1-x}]$$

$$H(M; x) = 1 + \frac{2x}{(1-x)^2} + \frac{2x}{1-x}$$