1. Generating functions of sequences which are eventually polynomial. For a function $f : \mathbb{N} \to \mathbb{N}$ prove that the following are equivalent:

   (a) There exists a polynomial $F(x)$ of degree d such that $f(n) = F(n)$ for all sufficiently large integers n.

   (b) There exists a polynomial $g(x)$ such that

$$\Sigma_{n\geq 0} f(n)x^n = \frac{g(x)}{(1-x)^{d+1}}.$$

Proof. ($\Rightarrow$): Suppose there exists a polynomial $F(x)$ of degree d such that $f(n) = F(n)$ for all sufficiently large integers n. Then we can write

$$\Sigma_{n\geq 0} f(n)x^n = f(0)+f(1)x+\cdots+f(n-1)x^{n-1}+F(n)x^n+\Sigma_{1\geq 0}F(n+i)x^{n+i}.$$

Claim: The series $\Sigma_{i\geq 1} i^d x^{i-1} = \frac{p(x)}{(1-x)^{d+1}}$. We prove the claim by induction: The case when $d = 1$ is done. Suppose it is true for $d = n$ Then

$$\Sigma_{i\geq 1} i^n x^{i-1} = 1 + 2^n x + 3^n x^2 + \cdots = \frac{p(x)}{(1-x)^{n+1}}$$

Multiplying this series by x and then differntiating term by term we have

$$\Sigma_{i\geq 1} i^{n+1} x^{i-1} = 1 + 2^{n+1} x + 3^{n+1} x^2 + \ldots$$

Therefore since $\frac{d}{dx}\frac{p(x)}{(1-x)^{n+1}} = \frac{p'(x)(1-x)+(n+1)p(x)}{(1-x)^{n+2}}$ we've proved the claim. By assumption $F(x) = a_d x^d + \cdots + a_1 x + a_0$. Then

$$F(n+i) = a_d(n+i)^d + \cdots + a_1(n+i) + a_0 = c_d i^d + \ldots c_1 i + c_0.$$

where each c_j doesn't involve i. We can rewrite $\Sigma_{i\geq 1} F(n+i)x^{n+i}$ as

$$\Sigma_{i\geq 1}(c_d i^d+\cdots+c_1 i+c_0)x^{n+i} = c_d x^n \Sigma_{i\geq 1} i^d x^{i-1}+\cdots+c_1 x^n \Sigma_{i\geq 1} i x^{i-1}+c_0 x^n \Sigma_{i\geq 1} x^{i-1}.$$

and by the claim we have:

$$\Sigma_{i\geq 1} F(n+i)x^{n+i} = \frac{c_d x^n p_d(x)}{(1-x)^{d+1}} + \cdots + \frac{c_1 x^n p_1(x)}{(1-x)^2} + \frac{c_0 x^n}{(1-x)}.$$

Then

$$\Sigma_{n\geq 0} f(n)x^n = f(0)+f(1)x+\cdots+F(n)x^n+\frac{c_d x^n p_d(x)}{(1-x)^{d+1}}+\cdots+\frac{c_1 x^n p_1(x)}{(1-x)^2}+\frac{c_0 x^n}{(1-x)}.$$

which can be written $\frac{g(x)}{(1-x)^{d+1}}$ by writting each of the terms above as a polynomial over $(1-x)^{d+1}$ and collecting terms.

($\Leftarrow$) The series

$$\frac{1}{(1-x)^{d+1}} = \Sigma_{i=0}(\left(\binom{d+1}{i}\right))x^i.$$

Since $((\binom{d+1}{i})) = \binom{d+i}{d})$ we can write the series in choose notation as $\Sigma_{i=0}\binom{d+i}{i}x^i$. Let

$$g(x) = a_k x^k + \cdots + a_1 x + a_0.$$

Then by multiplying and collecting terms of the same degree we have

$$\frac{g(x)}{(1-x)^{d+1}} = \Sigma_{i=0}(a_k\binom{d+i-k}{d} + \cdots + a_0\binom{d+i}{d}x^i$$

where the term $a_k\binom{d+i-k}{d}$ is zero for $i < k$. For each coefficient a_j of $g(x)$ define $p_{a_j}(i) = \binom{d+i-j}{d}$. Then

$$p_{a_j} = \frac{(i-j+d)!}{(d!)(i-j)!} = \frac{(i-j+d)(i-j+d-1)\ldots(i-j+1)}{d!} = c_d i^d + \cdots + c_1 i + c_0$$

and so $p_{a_j}(i)$ is a polynomial. Let $f(n)$ be the coeficients of the above series. Then for $n > k$ $f(n) = \Sigma_{j=0}^{k} p_{a_j}(n)$ is polynomial. $\square$