4.) Let $I = \langle x_1x_3, x_1x_4, x_2x_4\rangle$ in $\mathbb{F}[x_1, x_2, x_3, x_4]$, and let I_d be the $\mathbb{F}$-vector space of homogeneous polynomials of degree d in I.

Note that all monomials in $\mathbb{F}[x_1, x_2, x_3, x_4]$ are in I except those of the form $x_1^i x_2^j, x_2^i x_3^j, x_3^i x_4^j$ where $i, j \geq 0$. Basically, any monomial which is divisible by three of x_1, x_2, x_3, x_4 is divisible by one of the generators of I. On the other hand, monomials of the form $x_1^i x_2^j$, $x_2^i x_3^j$, or $x_3^i x_4^j$ where $i, j \geq 0$ are not in I.

To count these, there are $\binom{d+3}{3}$ monomials in 4 variables of degree d. There are $d+1$ monomials of the form $x_1^i x_2^j$ with $i + j = d$. If we count these for $x_2^i x_3^j, x_3^i x_4^j$ as well, then we get $3(d+1)$ monomials, but we have overcounted x_2^d and x_3^d once each. Putting this all together gives that the number of monic polynomials in I of degree d is $\binom{d+3}{3} - 3d - 1$. So

$$H_I(d) = dim_{\mathbb{F}} I_d = \binom{d+3}{3} - 3d - 1.$$

To compute $H(I; x)$, we will use the following identity:

$$\sum_{d \geq 0} \binom{d+n-1}{n-1} x^d = \sum_{d \geq 0} (\#\{\text{monomials of degree } d \text{ in } n \text{ variables}\}) x^d = \frac{1}{(1-x)^n}.$$

Then write $H(d) = \binom{d+3}{3} - 3\binom{d+1}{1} + 2\binom{d}{0}$. This gives that

$$\begin{aligned}\sum_{d\geq 0} H(d)x^d &= \sum_{d\geq 0} \binom{d+3}{3} x^d - 3\sum_{d\geq 0}\binom{d+1}{1} x^d + 2\sum_{d\geq 0}\binom{d}{0} x^d \\ &= \frac{1}{(1-t)^4} - \frac{3}{(1-t)^2} + \frac{2}{(1-t)}\end{aligned}$$