combinatorial commutative algebra

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homework three due: wed. mar. 4 (sf) / vie. 6 de mar. (bog)

Note. You are encouraged to work together on the homework, but you must state who you worked with. You *must* write your solutions independently and in your own words. If you are turning in your homework by email, please send it to `cca.acc.cca@gmail.com`.

1. Generating functions of sequences which are eventually polynomial. For a function $f : \mathbb{N} \to \mathbb{N}$ prove that the following are equivalent:
   (i) There exists a polynomial $F(x)$ of degree d such that $f(n) = F(n)$ for all sufficiently large integers n.
   (ii) There exists a polynomial $g(x)$ such that
   $$\sum_{n \geq 0} f(n)x^n = \frac{g(x)}{(1-x)^{d+1}}.$$
2. Polynomials with integer values at the integers. Let $T \subset \mathbb{R}[x]$ be the ring of real polynomials $P(x)$ such that $P(n)$ is an integer whenever n is an integer.
   (a) Prove that T is a free abelian group and the polynomials $\binom{x}{0}, \binom{x}{1}, \binom{x}{2}, \ldots$ form a basis for it, where
   $$\binom{x}{k} = \frac{x(x-1)\cdots(x-k+1)}{k!}.$$
   (Hint: Consider the linear operator $\Delta : T \to T$ given by $\Delta f(x) = f(x+1) - f(x)$.)
   (b) Is the ring T Noetherian?
3. A recurrence for Hilbert series of monomial ideals. Let $I = \langle m_1, \ldots, m_k \rangle$ be a monomial ideal in $R = \mathbb{C}[x_1, \ldots, x_n]$. Let
   $$I' = \langle m_1, \ldots, m_{k-1} \rangle, \qquad J = \left\langle \frac{m_1}{\gcd(m_1, m_k)}, \ldots, \frac{m_{k-1}}{\gcd(m_{k-1}, m_k)} \right\rangle.$$
   Prove that
   $$H_{R/I}(d) = H_{R/I'}(d) - H_{R/J}(d).$$
   (Hint. Find an appropriate exact sequence.)
4. Hilbert series of monomial ideals. Compute the Hilbert function and series for the ring
   $$\mathbb{C}[w, x, y, z] / \left(\langle w, x \rangle \cap \langle y, z \rangle \right).$$
5. Hilbert series. Compute the Hilbert function and series of the ring
   $$\mathbb{C}[w, x, y, z]/\langle wy - z^2, wx - yz, xz - y^2 \rangle.$$
6. (Bonus: 10 points) The SET problem. (Thanks to Diane Maclagan.) Let $\mathbb{F}_3 = \{0, 1, 2\}$ be the field of three elements. We say that points u, v, w in $\mathbb{F}_3^n$ form a SET if for each i $(1 \leq i \leq n)$, the coordinates u_i, v_i, w_i are either all equal or all distinct. Let $f(n)$ be the largest possible size of a subset of $\mathbb{F}_3^n$ containing no SETs.

   Let R be the polynomial ring in 3^n variables x_v $(v \in \mathbb{F}_3^n)$, and let I be the ideal generated by all monomials $x_u x_v x_w$ with $u + v + w = 0$.

   (a) Prove that $f(n) = \dim(R/I)$.
   (b) For $n = 2$, find a free resolution for R/I and compute $f(2)$.
   (c) What can you say about R/I and $f(n)$ for larger values of n?

   Feel free to use a computer algebra system in this question.