algebraic combinatorics

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homework two . due tuesday sep 23 at 11:59pm

Note. You are encouraged to work together on the homework, but you must state who you worked with **in each problem**. You must write your solutions independently and in your own words. (I recommend putting away the notes from your discussions with others, and reproducing the solutions by yourself.)

1. (Permutations with at most one peak.) Let π be a permutation of $[n]$. We say i is a *peak* of π if $\pi_{i-1} < \pi_i > \pi_{i+1}$. We say 1 is a peak if $\pi_1 > \pi_2$, and n is a peak if $\pi_n > \pi_{n-1}$.
   Find the number of permutations of $[n]$ with at most one peak.
2. (Practice with generating functions.) Use generating functions to simplify:
   (a) The sum of the first $n+1$ Fibonacci numbers: $F_0 + F_1 + \cdots + F_n$
   (b) $\displaystyle\sum_{k=0}^{n} (-1)^k \binom{m}{k}\binom{m}{n-k}$
   (c) $\displaystyle\sum_{k=0}^{n} k\binom{n}{k}$
3. (Cafecito) A programmer with an upcoming deadline eats very unhealthily. He eats five kinds of meals: breakfast, lunch, dinner, tea, and cafecito. His eating schedule is very erratic; the only rule is that he **must** have a cafecito after any breakfast, lunch, or dinner. Let m_n be the number of different sequences of n meals that he can have. (The last meal cannot be a cafecito.)
   In at most five lines, including a justification, compute the generating function for m_n.
4. (Counting words.) Let w_n be the number of words of n letters which use only the letters A, C, R, or T, and which have no two consecutive As. (The words do not have to use all the letters A, C, R, and T.)
   (a) Compute the generating function $W(x) = \displaystyle\sum_{n\geq 0} w_n x^n$.
   (b) (Bonus, worth k points.) Write one such word of length k that actually exists in the language of your choice.
5. (Domino tilings of $3\times n$ rectangles.) Let t_n be the number of domino tilings of a $3\times n$ rectangle.
   (a) Find the generating function $T(x) = \displaystyle\sum_{n\geq 0} t_n x^n$.
   (b) Find a recurrence formula for t_n.
   (c) Find an asymptotic formula for t_n.
6. (Bonus problem 1.) We have n light bulbs in a row. Initially they are all off, and we wish to turn them all on. The first bulb can always be turned on or off. For each $i > 1$, the ith bulb can be switched (turned on or off) only when bulb $i-1$ is on and all earlier bulbs are off. Find the minimum number of switches needed to turn all bulbs on.
7. (Bonus problem 2.) Why does the number

   $$\phi = \frac{9}{10}\frac{99}{100}\frac{999}{1000}\frac{9999}{10000}\cdots = 0.89001009999899900000010000999999998999990000000000010\ldots$$

   have all its decimal digits equal to $0, 1, 8,$ or 9?