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homework one . due tuesday sep 9 at 11:59pm

Note. You are encouraged to work together on the homework, but you must state who you worked with **in each problem**. You must write your solutions independently and in your own words. (I recommend putting away the notes from your discussions with others, and reproducing the solutions by yourself.)

1. (Poker hands.) In a normal deck of 52 cards with no jokers, there are $\binom{52}{5} = 2'598.260$ possible hands of 5 cards. Find the number of hands which form:

   (a) a royal flush

   (b) a straight flush

   (c) four of a kind

   (d) a full house

   (e) a flush

   (f) a straight

   (g) three of a kind

   (h) two pairs

   (i) one pair

   Your answers should explain why poker hands are ranked in this order. (Count the hands in a way that the categories don't overlap. For example, don't count a "three of a kind" in the "one pair" category..)

   Note. See the second page for definitions. (The spelling mistake is not mine.)

2. (Practice with generating functions.) Find the generating functions for the following sequences:

   (a) $a_n = 2n - 1$

   (b) $b_n = 3^{n-1}$

   (c) $c_0 = 1$ and $c_{n+1} = 2c_n + 3$ for $n \geq 0$

   (d) $d_0 = 0, d_1 = 1$ and $d_{n+1} = 5d_n - 6d_{n-1}$ for $n \geq 0$

3. (A combinatorial identity.) Prove combinatorially that $\displaystyle\sum_{k=0}^{m} \binom{n}{k}\binom{n-k}{m-k} = 2^m \binom{n}{m}$.

4. (A combinatorial identity.) Prove algebraically that $\displaystyle\sum_{k=0}^{m} \binom{n}{k}\binom{n-k}{m-k} = 2^m \binom{n}{m}$.

5. (Cafecito) A programmer with an upcoming deadline eats very unhealthily. He eats five kinds of meals: breakfast, lunch, dinner, tea, and cafecito. His eating schedule is very erratic; the only rule is that he **must** have a cafecito after any breakfast, lunch, or dinner.

   (a) How many different sequences of n meals can he have? (As stipulated, the last meal cannot be breakfast, lunch, or dinner.)

   (b) Find the expected proportion of cafecitos the programmer has in the long run.

   (More concretely, let c_n be the average number of cafecitos among all such sequences of n meals. Find $\lim_{n\to\infty} c_n/n$.)

6. (Bonus problem 1.) Why does the number

$$\phi = \frac{9}{10}\frac{99}{100}\frac{999}{1000}\frac{9999}{10000}\cdots = 0.89001009999899990000000100009999999998999990000000000010\ldots$$

have all its decimal digits equal to $0, 1, 8$, or 9?

7. (Bonus problem 2.) Find the number of ways to take an n element set S, and, if S has more than one element, to partition S into two disjoint nonempty subsets; then take one of these two sets with more than one element and partition it into two disjoint nonempty subsets; then take one of the remaining sets with more than one element and partition it into two disjoint nonempty subsets, etc., until only one-element subsets remain.

   For example, we could start with 12345678 (short for $\{1, 2, 3, 4, 5, 6, 7, 8\}$), then partition it into 126 and 34578, then partition 34578 into 4 and 3578, then 126 into 6 and 12, then 3578 into 37 and 58, then 58 into 5 and 8, then 12 into 1 and 2, and finally 37 into 3 and 7.

   (Note: The order in which we partition the sets is important; for instance, partitioning 1234 into 12 and 34, then 12 into 1 and 2, and then 34 into 3 and 4, is different from partitioning 1234 into 12 and 34, then 34 into 3 and 4, and then 12 into 1 and 2. However, partitioning 1234 into 12 and 34 is the same as partitioning it into 34 and 12.)