TROPICAL HYPERPLANE ARRANGEMENTS AND ORIENTED MATROIDS

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Abstract. We study the combinatorial properties of a tropical hyperplane arrangement. We define tropical oriented matroids, and prove that they share many of the properties of ordinary oriented matroids. We show that a tropical oriented matroid determines a subdivision of a product of two simplices, and conjecture that this correspondence is a bijection.

1. Introduction

Tropical mathematics is the study of the tropical semiring consisting of the real numbers with the operations of $+$ and max. This semiring can be thought of as the image of a power series ring under the degree map which sends a power series to its leading exponent. This semiring has received great attention recently in several areas of mathematics, due to the discovery that there are often strong relationships between a classical question and its tropical counterpart. One can then translate geometric questions about algebraic varieties into combinatorial questions about polyhedral fans. This point of view has been fruitful in algebraic geometry, combinatorics, and phylogenetics, among others [2, 3, 6, 7, 11, 16, 22].

The triangulations of a product of two simplices are ubiquitous and useful objects. They are of independent interest [4, 5, 9, 13], and have been used as a building block for finding efficient triangulations of high dimensional cubes [10, 12] and disconnected flip-graphs [18, 19]. They also arise very naturally in connection with the Schubert calculus [1], Hom-complexes [14], growth series of root lattices [21], transportation problems and Segre embeddings [23], among others.

The goal of this paper is to start laying down the foundations of a theory of tropical oriented matroids. In the same way that oriented matroids capture the combinatorial properties of real hyperplane arrangements and ordinary polytopes, these objects are modeled after tropical hyperplane arrangements and tropical polytopes. We present strong evidence of the intimate connection between them and the subdivisions of a product of two simplices: a tropical oriented matroid determines a subdivision, and we conjecture that this is a bijection. We expect that further development of the theory will lead to elegant structural results and applications in the numerous areas where these objects appear; we present several results and conjectures to that effect.

The paper is organized as follows. In Section 2 we recall some background information on tropical geometry. Section 3 defines tropical oriented matroids, and proves that every tropical hyperplane arrangement gives rise to one. In Section 4 we prove that a tropical oriented matroid is completely determined by its topes (maximal elements), and is also determined by its vertices (minimal elements). We define the notions of deletion and contraction. Section 5 presents three key conjectures for tropical oriented matroids: a bijection with subdivisions of a product of two simplices, a notion of duality, and a topological representation theorem. We then show potential applications

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to three open problems in the literature. Finally, Section 6 shows that a tropical oriented matroid does determine such a subdivision, and proves the reverse direction for triangulations in the two-dimensional case.

2. Basic definitions

In this section we recall some basic definitions from tropical geometry. For more information, see [7, 22].

Definition 2.1. *The **tropical semiring** is given by the real numbers $\mathbb{R}$ together with the operations of tropical addition $\oplus$ and tropical multiplication $\odot$ defined by $a \oplus b = max(a, b)$ and $a \odot b = a + b$.*

This tropical semiring can be thought of as the image of ordinary arithmetic in a power series ring under the degree map, which sends a power series in t^{-1} to its leading exponent. As in ordinary geometry, we can form tropical d-space, $\mathbb{R}^d$ with the operations of vector addition (coordinatewise maximum) and scalar multiplication (adding a constant to each vector.) For many purposes, it proves convenient to work in tropical projective $(d-1)$-space $\mathbb{TP}^{d-1}$, given by modding out by tropical scalar multiplication; this produces the ordinary vector space quotient $\mathbb{R}^d/(1, \ldots, 1)\mathbb{R}$, which can be depicted as real $(d-1)$-space.

In this space, tropical hyperplanes form an important class of objects. These are given by the vanishing locus of a single linear functional $\bigoplus c_i \odot x_i$; in tropical mathematics, this vanishing locus is defined to be the set of points where the encoded maximum $\max(c_1 + x_1, \ldots, c_d + x_d)$ is achieved at least twice. (Reflection on the power series etymology of tropical mathematics will yield the motivation for this definition.)

These tropical hyperplanes are given by fans polar to the simplex formed by the standard basis vectors $\{e_1, \ldots, e_d\}$; the apex of $\bigoplus c_i \odot x_i$ is $(-c_1, \ldots, -c_d)$. Each of these fans has a natural index on each of its cones: the subset of $[d]$ for which $c_i + x_i$ is maximized. On the d full-dimensional sectors, this is a singleton; on the cone polar to a subset of basis vectors forming a face of the simplex, it is given by that subset of coordinates. Figure 1 shows what tropical hyperplanes look like in $\mathbb{TP}^2$, where the point (a, b, c) in $\mathbb{TP}^2$ is represented by the point $(0, b-a, c-a)$ in $\mathbb{R}^2$.

Another natural class of geometric objects in tropical mathematics is that of tropical polytopes.

Definition 2.2. *Given a set of points $V = \{v_1, \ldots, v_n\} \subset \mathbb{TP}^{d-1}$, their **tropical convex hull** is the set of all (tropical) linear combinations[1] $\bigoplus c_i \odot v_i$ with $c_i \in \mathbb{R}$, where the scalar multiplication $c_i \odot v_i$ is defined componentwise. A **tropical polytope** is the tropical convex hull of a finite set of points.*

Tropical polytopes are bounded polyhedral complexes [7]. An important theorem connects tropical polytopes with tropical hyperplane arrangements:

Theorem 2.3. [7] *Let P be the tropical convex hull of a finite point set $V = \{v_1, \ldots, v_n\}$. Then P is the union of the bounded regions of the polyhedral decomposition of $\mathbb{TP}^{d-1}$ given by putting an inverted hyperplane at each point $v_1, \ldots, v_n$.*

This inverted hyperplane arrangement is of course combinatorially equivalent to a hyperplane arrangement given by hyperplanes with apexes $\{-v_1, \ldots, -v_n\}$. Indeed, both of these are equivalent to the regular subdivisions of a product of simplices; see [23, 24] for more information on this topic.

[1] Note that we take **all** linear combinations, not just the nonnegative ones adding up to 1 as in regular convexity.

Theorem 2.4. [7] *The convex hull of a finite point set $V = \{v_1, \ldots, v_n\}$, with the polyhedral subdivision given by Theorem 2.3 is combinatorially isomorphic to the complex of interior faces of the regular subdivision of $\Delta^{n-1} \times \Delta^{d-1}$, where the height of vertex (i, j) is given by the j-th coordinate of v_i. The corresponding tropical hyperplane arrangement is isomorphic to the complex of faces which contain at least one vertex from each of the n copies of Δ^{d-1}.*

Indeed, we can say precisely which face each region of the hyperplane arrangement corresponds to; see the discussion of Section 5.

The combinatorial structure of a tropical hyperplane arrangement is captured by its collection of **types**. Given an arrangement $H_1, \ldots, H_n$ in $\mathbb{TP}^{d-1}$, the **type** of a point $x \in \mathbb{TP}^{d-1}$ is the n-tuple $(A_1, \ldots, A_n)$, where $A_i \subseteq [d]$ is the set of closed sectors of the hyperplane H_i which x is contained in.[2] Algebraically, if hyperplane H_i has vertex $v_i = (v_{i1}, \ldots, v_{id})$, the set A_i consists of the indices j for which $x_j - v_{ij}$ is maximal.

Since all the points in a face of the arrangement have the same type, we call this the type of the face. Figure 1 shows an arrangement of three tropical hyperplane arrangements in $\mathbb{TP}^2$, and indicates the types of some of the faces. The labelling of the sectors of the hyperplanes is indicated on the right.

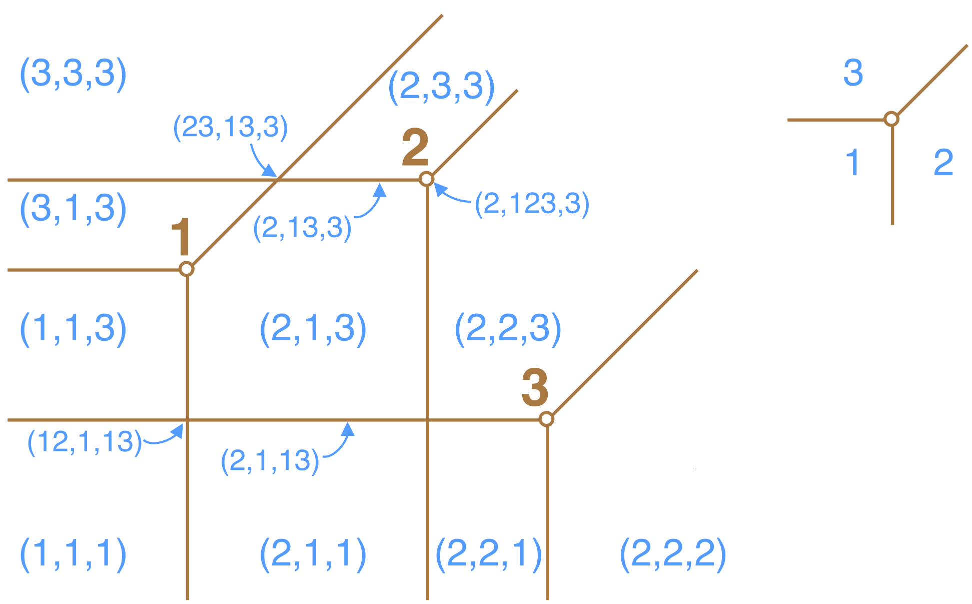

FIGURE 1.
An arrangement of three hyperplanes in $\mathbb{TP}^2$ and the types of some of its faces.

3. Tropical oriented matroids.

Our goal will be to define tropical oriented matroids as a collection of types (analogous to the covectors of an ordinary oriented matroid) which satisfy certain axioms inspired by tropical hyperplane arrangements as well as by ordinary oriented matroids. We proceed with some preliminary definitions.

[2]Notice that this definition of type is the transpose of the definition in [7].

Definition 3.1. *An (n,d)-**type** is an n-tuple $A = (A_1, \ldots, A_n)$ of nonempty subsets of $[d] := \{1, \ldots, d\}$. We will call $A_1, \ldots, A_n$ the **coordinates** of A, $1, \ldots, n$ the **positions**, and $1, \ldots, d$ the **directions**.*

We will often encode an (n,d)-type as a subgraph of the complete bipartite graph $K_{n,d}$, where edge ij (with $1 \leq i \leq n$ and $1 \leq j \leq d$) is in the subgraph if and only if $j \in A_i$.

Definition 3.2. *Given two (n,d)-types A and B, the **comparability graph** $CG_{A,B}$ has vertex set $[d]$. For $1 \leq i \leq n$, we draw an edge between j and k for each $j \in A_i$ and $k \in B_i$. That edge is undirected if $j, k \in A_i \cap B_i$, and it is directed $j \to k$ otherwise.*

This object is a **semidigraph**:

Definition 3.3. *A **semidigraph** is a graph with some undirected edges and some directed edges. A **directed path** from a to b in a semidigraph is a collection of vertices $v_0 = a, v_1, \ldots, v_k = b$ and a collection of edges $e_1, \ldots, e_k$, at least one of which is directed, such that e_i is either a directed edge from v_{i-1} to v_i or an undirected edge connecting the two. A **directed cycle** is a directed path with identical endpoints. A semidigraph is **acyclic** if it has no directed cycles.*

A property that a point lying in a tropical hyperplane arrangement should have is that its type should locally change in a predictable way. The next definition will help us rigorize this.

Definition 3.4. *The **refinement** of a type $A = (A_1, \ldots, A_n)$ with respect to an ordered partition $P = (P_1, \ldots, P_r)$ of $[d]$ is $A_P = (A_1 \cap P_{m(1)}, \ldots, A_n \cap P_{m(n)})$ where $m(i)$ is the largest index for which $A_i \cap P_{m(i)}$ is non-empty. A refinement A_P is **total** if all of its entries are singletons.*

With these definitions, we are ready to give a natural axiomatic definition of a tropical oriented matroid.

Definition 3.5. *A **tropical oriented matroid** M (with parameters (n,d)) is a collection of (n,d)-types which satisfy the following four axioms:*

- *Boundary: For each $j \in [d]$, the type $\mathbf{j} := (j, j, \ldots, j)$ is in M.*
- *Elimination: If we have two types A and B in M and a position $j \in [n]$, then there exists a type C in M with $C_j = A_j \cup B_j$, and $C_k \in \{A_k, B_k, A_k \cup B_k\}$ for all $k \in [n]$.*
- *Comparability: The comparability graph $CG_{A,B}$ of any two types A and B in M is acyclic.*
- *Surrounding: If A is a type in M, then any refinement of A is also in M.*

The boundary axiom is illustrated in Figure 1. Tropical hyperplanes are all translates of each other; if we go off to infinity in a given basis direction, then we end up in that sector of each hyperplane.

The elimination axiom roughly tells us how to go from a more general to a more special position with respect to a hyperplane. It predicts what happens when we intersect the jth hyperplane as we walk from A to B along a tropical line segment. This is illustrated in the left panel of Figure 2. For ordinary oriented matroids, if we have two covectors which have opposite signs in some position, then we can produce a covector which has a 0 in that position and agrees with A and B whenever these two agree. Our elimination axiom is just the d-sign version of this; if we replace the set $[d]$ with the signs $\{+, -\}$, using the 2-element set $\{+, -\}$ for 0, then the elimination axiom yields the normal oriented matroid elimination axiom. Note that a type entry A_i being larger corresponds to it being more like 0; the ultimate vanishing point in the hyperplane is its apex v_i, where $A_i = [d]$.

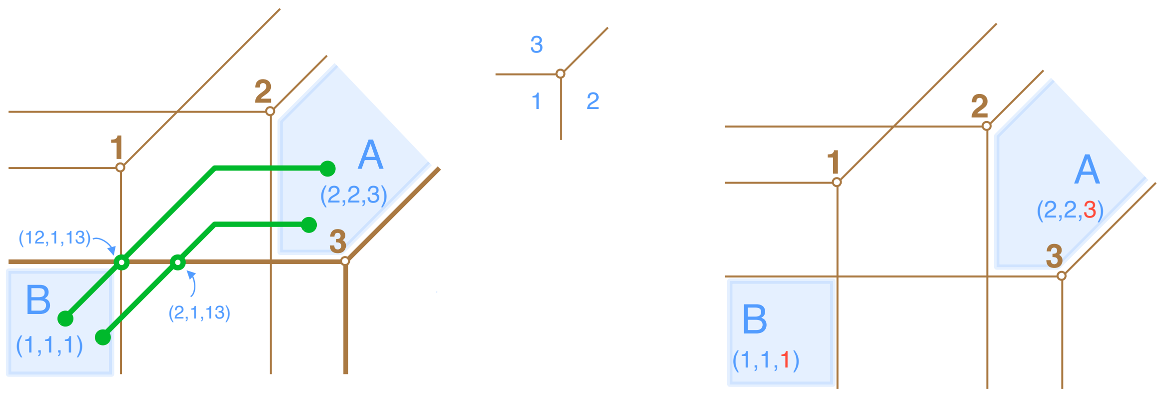

FIGURE 2.
Elimination: Walking from A to B, one must cross hyperplane 3.
Comparability: Walking from A to B, one must move more in the 1 direction than in the 3 direction. This gives an arrow $3 \to 1$ in $CG_{A,B}$.

As illustrated in the right panel of Figure 2, an edge from j to k in the comparability graph $CG_{A,B}$ roughly indicates that, as one walks from region A to region B, one moves more in the k direction than in the j direction. Therefore this graph cannot have cycles.

The surrounding axiom tells us how to go from a more special to a more general position. In ordinary oriented matroids, for any two covectors A and B, we can find a covector which agrees with A everywhere except where the sign of A is 0, when it agrees with the sign of B. This corresponds to moving infinitesimally from A towards B; essentially, doing this process for all B yields local information about signs around A, where B-covectors are proxies for directions. Because tropical hyperplanes are all translates of each other, we don't need any directional information from B; refining a type corresponds to moving infinitesimally away from the point in a direction indicated by an ordered partition. In other words, we can write down the types of points in a local neighborhood of A simply by looking at A itself, which is what the surrounding axiom does.

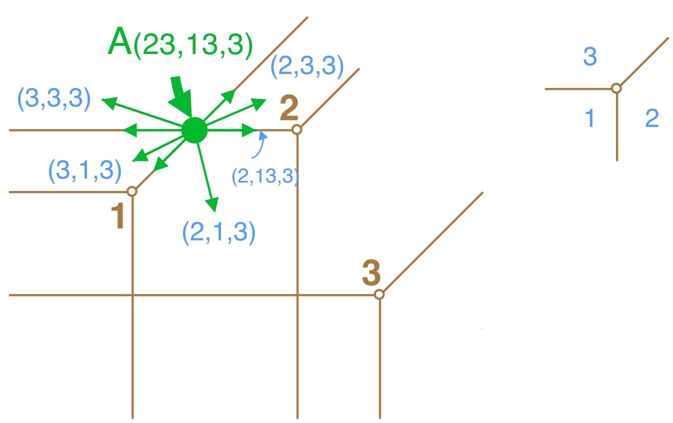

FIGURE 3.
Surrounding: The types surrounding A are obtained by refining A with respect to the ordered partitions of [3].

Obviously, if this definition is to have any merit, the following must be true.

Theorem 3.6. *The collection of types in a tropical hyperplane arrangement forms a tropical oriented matroid.*

Proof. Let the apexes of the arrangement be given by $\{v_1, \ldots, v_n\}$. We verify the axioms in the order given above.

Boundary: Taking a point with x_j large enough (specifically, such that $x_j - x_i > v_{kj} - v_{ki}$ for all $k \in [n]$ and $i \neq j \in [d]$) works.

Elimination: Suppose we have points x and y of types A and B, and a position $j \in [n]$. Let $a \in A_j$ and $b \in B_j$, and pick coordinates for x and y (by adding a multiple of $(1, \ldots, 1)$) such that $x_a - v_{ja} = y_b - v_{jb} = 0$. Consider the point $z = x \oplus y$, the coordinatewise maximum of x and y. Then $x_i - v_{ji} = 0$ for $i \in A_j$, and this maximizes this difference over all i; similarly, $y_i - v_{ji} = 0$ for $i \in B_j$, and this maximizes this difference over all i. Therefore $z_i - v_{ji}$ is 0 for $i \in A_j \cup B_j$, and it is negative for other i. Let C be the type of z; then $C_j = A_j \cup B_j$.

Now, consider any $k \neq j$. We have $z_i - v_{ki} = \max(x_i - v_{ki}, y_i - v_{ki})$; we need to find the values of i for which this is maximized. The maximum value of this is equal to $\max(\max_i(x_i - v_{ki}), \max_i(y_i - v_{ki}))$. If the first maximum is bigger, then the values of i which maximize $z_i - v_{ki}$ are precisely those which maximize $x_i - v_{ki}$, *i.e.* $C_k = A_k$. Similarly, if the second maximum is bigger, then we have $C_k = B_k$. Finally, if the maxima are the same, then we have $C_k = A_k \cup B_k$.

Essentially, we have formed the tropical line segment between x and y, and taken the point on it which is on the maximal cone of H_j. Every point on this line segment has $C_k \in \{A_k, B_k, A_k \cup B_k\}$ for all k.

Comparability: Let A and B be types realized by points x and y. An edge from i to j in $CG_{A,B}$ indicates that, in some position p, A_p contains i and B_p contains j. This gives $x_i - v_{pi} \geq x_j - v_{pj}$ and $y_j - v_{pj} \geq y_i - v_{pi}$, which implies $x_i - y_i \geq x_j - y_j$. If the edge is directed, then one of these two inequalities is strict, and $x_i - y_i > x_j - y_j$. Intuitively, if we walk from A to B, we move more in the i direction than in the j direction. Therefore a cycle in $CG_{A,B}$ would give a series of inequalities which add up to $0 > 0$.

Surrounding: Take a point x with type A, and an ordered partition $P = (P_1, \ldots, P_r)$. Let $\Delta x = \epsilon(f(1), \ldots, f(d))$, where $f(i)$ equals the value of j such that $i \in P_j$. We claim that for ϵ sufficiently small, $x + \Delta x$ has type A_P. Take any index k. Then the elements of the k-th coordinate of the type of $x + \Delta x$ are those i which maximize $(x + \Delta x)_i - v_{ki}$. Since ϵ is sufficiently small, the only way this can happen is if $x_i - v_{ki}$ is maximized for this i (*i.e.* $i \in A_k$). Among these elements, $(x + \Delta x)_i - v_{ki}$ is maximized if and only if $\Delta x_i = \epsilon f(i)$ is maximized, *i.e.* if i is maximal among A_k with respect to P. This completes the proof. □

4. Properties of tropical oriented matroids

The following definitions are motivated by ordinary oriented matroids.

Definition 4.1. *Given a type A, consider the associated undirected graph G_A with vertex set $[d]$ which is given by connecting i and j if there exists some coordinate A_k with $i, j \in A_k$. The **dimension** of a type A is given by the number of connected components of G_A, minus one. A **vertex** of a tropical oriented matroid is a type A with G_A connected;* i.e., *one of dimension 0. A **tope** is a type $A = (A_1, \ldots, A_n)$ such that each A_i is a singleton;* i.e., *one of full dimension $d - 1$.*

For tropical hyperplane arrangements, the dimension of a type, as defined above, equals the dimension of the region it describes. [7, Proposition 17]

The following lemma will be useful.

Lemma 4.2. *Refinement is transitive: if C is a refinement of B, and B is a refinement of A, then C is a refinement of A.*

Proof. Suppose B is a refinement of A with respect to the ordered partition $(P_1, \ldots, P_r)$, and C is a refinement of B with respect to the ordered partition $(Q_1, \ldots, Q_s)$. Let $X_{ij} = P_i \cap Q_j$. Then it is easy to see that C is the refinement of A given by $(X_{11}, X_{12}, \ldots, X_{1s}, X_{21}, \ldots, X_{rs})$. □

Lemma 4.3. *Suppose that A and B are types of a tropical oriented matroid, and suppose that we have $B_i \subseteq A_i$ for all $i \in [n]$. Then B is a refinement of A.*

Proof. Suppose that B is not a refinement of A. This means that there is no way to consistently break ties among each A_i so that B_i consists of the maximal elements of A_i; in other words, the set of equations given by $x_j = x_k$ for $j, k \in B_i$ and $x_j > x_k$ for $j \in B_i, k \in A_i \setminus B_i$ has no solution. By linear programming duality, this implies that some linear combination of these adds up to $0 > 0$. The inequalities which contribute will then form a directed cycle in the comparability graph $CG_{A,B}$, violating the comparability axiom. □

As in ordinary oriented matroids, we have the following theorem.

Theorem 4.4. *The topes of a tropical oriented matroid M completely determine it. More precisely, a type A is in M if and only if the following two conditions hold:*

- *A satisfies the compatibility axiom with every tope of M (*i.e.* $CG_{A,T}$ is acyclic for every tope T of M.)*
- *All of A's total refinements are topes of M.*

Proof. First, note that if A satisfies the conditions, so does every refinement B of A: the total refinements of B form a subset of the total refinements of A, and the comparability graph $CG_{B,T}$ is a subgraph of the comparability graph $CG_{A,T}$ for each T.

Suppose that we have a minimal counterexample with respect to refinement: an n-tuple A such that every refinement of A is in M, but A itself is not. Throughout the following argument, it will be useful to keep in mind the example $A = (12, 69, 12, 67, 23, 18, 345, 135)$.

We know that A has some element that is not a singleton; without loss of generality, assume that A_1 contains $\{1, 2\}$. Consider the connected components of $G_A \setminus 1$, the graph obtained from G_A by deleting vertex 1 and the edges incident to it. One of these components contains 2; let S consist of this subset of $[d]$, without loss of generality $\{2, \ldots, r\}$, and let T be $\{r+1, \ldots, d\}$. In the example, $S = \{2, 3, 4, 5\}$ and $T = \{6, 7, 8, 9\}$.

Now consider the refinements B and C of A given by the ordered partitions $(S, 1 \cup T)$ and $(1 \cup T, S)$, respectively. In our example $B = (1, 69, 1, 67, 23, 18, 345, 1)$ and $C = (2, 69, 2, 67, 23, 18, 345, 35)$. Since $B_1, C_1 \neq A_1$, we know that B and C are proper refinements of A, and therefore are in M by the minimality assumption on A.

Now eliminate in position 1 between B and C to get some element D of M. In our example we have $D = (12, 69, *, 67, 23, 18, 345, *)$. We will prove that $D = A$, thereby showing that A is in M. In the first position we have $D_1 = B_1 \cup C_1 = 1 \cup (A_1 \setminus 1) = A_1$. In other positions i we have $D_i \in \{B_i, C_i, B_i \cup C_i\}$. Notice that A_i cannot contain elements in S and T simultaneously.

If $A_i \subseteq S$ or $A_i \subseteq 1 \cup T$ then we have $B_i = C_i = A_i$ so $D_i = A_i$. The remaining case is when $A_i = 1 \cup X$ for $X \subseteq S$; here $B_i = 1$ and $C_i = X$. The proof of the theorem is then complete with the following lemma. □

Lemma 4.5. *In the situation of the remaining case of Theorem 4.4, where $A_i = 1 \cup X$ for $X \subseteq S$, $B_i = 1$, and $C_i = X$, we have $D_i = 1 \cup X = A_i$.*

Proof. Consider those i for which $D_i = 1$ or $D_i = C_i$. Since each element x of C_i is in S, there exists a path P in G_A connecting 2 (which is in S) to x. Over all such triples (i, x, P), choose one such that P has minimum length. In our example, we might have $i = 8$, $x = 3$, and the path $P = 23$; we would then know that $D = (12, 69, 12, 67, 23, 18, 345, *)$

For notational convenience, assume that the path from 2 to x is $23 \ldots x$. This means that there exist positions in A containing each of $\{1, 2\}, \ldots, \{x-1, x\}$ and $\{1, x\}$. In our example, we are talking about the first, fifth, and eight positions. In other words, if we denote by A' and D' the restrictions of A and D to these positions and to the numbers $1, \ldots, x$, then $A' = (12, 23, \ldots, (x-1)x, 1x)$. By the minimality of P, none of the numbers $\{1, 2, \ldots, x\}$ aside from the given ones appear in any of these positions. Also by minimality, D' agrees with A' in all of these positions, except the last one (which is the ith position), where D'_i is $\{1\}$ or $\{x\}$.

Suppose $D'_i = \{1\}$, so that $D' = (12, 23, \ldots, (x-1)x, 1)$. Consider the tope T one obtains by refining D by the partition $(\{d\}, \{d-1\}, \ldots, \{x+1\}, \{1\}, \{2\}, \ldots, \{x\})$. In these positions, T is $(2, 3, \ldots, x, 1)$. But this tope is incomparable to A; $CG_{A,T}$ contains the directed cycle $(1, 2, \ldots, x)$. This is a contradiction.

Similarly, suppose that $D'_i = \{x\}$, so that $D' = (12, \ldots, (x-1)x, x)$. When we refine D by $(\{d\}, \{d-1\}, \ldots, \{x+1\}, \{x\}, \{x-1\}, \ldots, \{2\}, \{1\})$ we obtain a tope U with $(1, 2, \ldots, x-1, x)$ in these positions. This tope is also incomparable to A, with $CG_{A,U}$ containing the same cycle in the opposite direction.

We have reached a contradiction, which settles the desired result. □

The vertices also determine the tropical oriented matroid.

Theorem 4.6. *A tropical oriented matroid is completely determined by its vertices. More precisely, the types of M are the refinements of its vertices.*

Proof. We need to show that if a type $A = (A_1, \ldots, A_n)$ is not a vertex, then there exists some type of which it is a refinement. By Lemma 4.3, we just need to find a type which strictly contains it. To simplify notation, we will do a proof by example; this method clearly works in general.

Suppose that A is not a vertex. This means that the graph G_A is disconnected. There are two possible cases:

Case 1. All of the numbers appearing in the A_i are in the same connected component, but there is an element of $[d]$ which appears in none of them. Suppose $A = (123, 14, 24, 234, 23)$. Eliminate A with $\mathbf{5} = (5, 5, 5, 5, 5)$ in position 1. This yields $B = (1235, B_2, B_3, B_4, B_5)$, where each B_i is equal to A_i, $\{5\}$, or $A_i \cup \{5\}$. We are done unless some B_i is equal to $\{5\}$. In this case, we eliminate B with A in that position to get a type C: then C contains A in each coordinate where B does, as well as in the position we just eliminated. Continue this process, eliminating in each position where the resulting type is equal to $\{5\}$, until we obtain a type which contains A in every coordinate. In the last position we eliminated in, this type also contains $\{5\}$, so it strictly contains A. This completes the proof.

Case 2. The numbers appearing in the A_i are in two or more connected components of G_A. Take the following example:

$$A = (12, 46, 256, 135, 34, 78, 79, 9, 7)$$

so that one connected component of G_A is $[6]$. In particular, every entry of A is either a subset of $[6]$ or of $[7, 9]$. We use the symbol S^* to represent a non-empty set containing only elements from S. The graph G_A is shown in Figure 4. In our example, the induced subgraph of G_A with vertices $1, 2, \ldots, i$ is connected for all $1 \leq i \leq 6$. We will need this property, which can be accomplished in the general case by suitable relabelling.

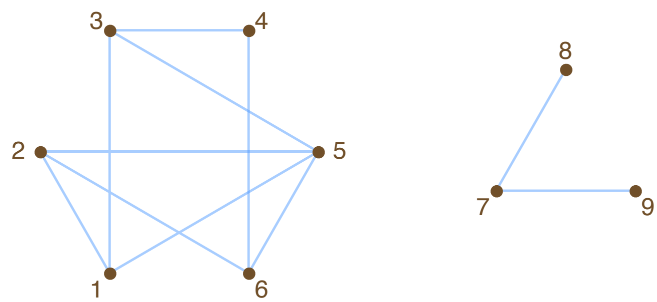

FIGURE 4.
The graph G_A for $A = (12, 46, 256, 135, 34, 78, 79, 9, 7)$.

We claim first that there is a type of the form

$$A_6 = (12, 46, 256, 135, 34, [6]^*, [6]^*, [6]^*, [6]^*);$$

i.e., a type which agrees with A on all coordinates contributing to the first component of A, and consists only of elements from $[6]$. We will build it up one step at a time by building a type A_i $(1 \leq i \leq 6)$ with elements from $[i]$ such that, wherever the restriction of A to $[i]$ is nonempty, it agrees with A_i.

First, eliminate between **1** and **2** in position 1. This yields a type

$$A_2 = (12, [2]^*, [2]^*, [2]^*, [2]^*, [2]^*, [2]^*, [2]^*, [2]^*, [2]^*).$$

But in any position where A contains a 1 or a 2 (or both), A_2 agrees with $A \cap [2]$; if not, then the acyclic comparability graph $CG(A_2, A)$ has a directed edge between elements of $[2]$, which is impossible as it also has an undirected edge between 1 and 2 from the first position. So

$$A_2 = (12, [2]^*, 2, 1, [2]^*, [2]^*, [2]^*, [2]^*, [2]^*).$$

To build A_3 we eliminate between A_2 and **3** in position 4, to get

$$A_3 = ([3]^*, [3]^*, [3]^*, 13, [3]^*, [3]^*, [3]^*, [3]^*, [3]^*).$$

Now, in position 1, A_3 equals 3, 12 or 123; and in fact, it has to equal 12, since $CG(A_3, A)$ already has an undirected edge between 3 and 1. We now see that $\{1, 2, 3\}$ is connected by undirected edges in $CG(A_3, A)$. Therefore, wherever A contains a 1,2,or 3, A_3 must agree with $A \cap [3]$; that is,

$$A_3 = (12, [3]^*, 2, 13, 3, [3]^*, [3]^*, [3]^*, [3]^*).$$

We continue in this way. To build A_i, we eliminate between A_{i-1} and A in a position which connects i to $\{1, \ldots, i-1\}$ in G_A. We then "grow" a spanning tree of G_A restricted to $[i]$, starting at vertex

i. Each edge in the tree guarantees that A_i and $A \cap [i]$ agree in a new coordinate. Once we have done this, we know that all of $[i]$ is connected in $CG(A_i, A)$ by undirected edges. Thus, wherever A contains a $1, 2, \ldots,$ or i, A_i must agree with $A \cap [i]$.

In our example, we obtain A_4 by eliminating between A_3 and A in position 5. We grow the spanning tree with edges 43, 31, 12 in that order. This guarantees, in that order, that position 5 is 34, position 4 is 13, and position 1 is 12. This means that [4] is connected in $CG(A_4, A)$ by undirected edges, which forces position 2 to be 4 and position 3 to be 2. Therefore

$$A_4 = (12, 4, 2, 13, 34, [4]^*, [4]^*, [4]^*, [4]^*).$$

We eliminate again with **5** in position 3, and then with **6** in position 3, to obtain the desired type.

Now that we have a type

$$A_6 = (12, 46, 256, 135, 34, [6]^*, [6]^*, [6]^*, [6]^*),$$

we proceed as in case 1. We eliminate A_6 with A in position 6 to obtain an element which contains A in the first seven positions, and strictly contains A (in position 6.) If there are any positions where the new type does not contain A, it consists there of a subset of $[6]^*$; we eliminate in that position with A. We continue doing this, eventually obtaining an element which contains A in every position and strictly contains A in the position where we last eliminated. □

The next two propositions establish the tropical analogues of the standard matroid operations of deletion and contraction.

Proposition 4.7. *Let M be a tropical oriented matroid with parameters (n, d). Pick any coordinate $i \in [n]$. Then the* ***deletion*** *$M_{\backslash i}$, which consists of all $(n-1, d)$ types which arise from types of M by deleting coordinate i, is also a tropical oriented matroid.*

Proof. It is straightforward to verify each axiom. For boundary, the deletion of $\mathbf{j} = (j, j, \ldots, j)$ is again $(j, j, \ldots, j)$. To eliminate between two types of $M_{\backslash i}$, simply find any preimages of them in M and eliminate between these. The comparability graph $CG_{A,B}$ is a subgraph of the comparability graph of any preimages of A and B in M, and hence is acyclic. Finally, the surrounding axiom holds, as refinement commutes with deletion. □

Proposition 4.8. *Let M be a tropical oriented matroid with parameters (n, d). Pick any direction $i \in [d]$. Then the* ***contraction*** *$M_{/i}$, which consists of all types of M which do not contain i in any coordinate, is also a tropical oriented matroid (with parameters $(n, d-1)$.)*

Proof. Again, it is straightforward to verify each axiom; assume that $i = d$ for notational convenience. For $j \in [d-1]$, $\mathbf{j}$ is a type of M not containing d, and hence is a type of $M_{/d}$, so the boundary axiom holds. When we eliminate between two types, no new symbols are introduced, so the elimination of two types of $M_{/d}$ in M is again a type of M not containing d, and hence is in $M_{/d}$. The comparability graph of two types in $M_{/d}$ is the same as their comparability graph in M, except for the isolated vertex d (which is removed), and is therefore also acyclic. The surrounding axiom is trivial, as the process of refinement does not introduce any new symbols, and hence any refinement in M of a type in $M_{/d}$ is again in $M_{/d}$. □

5. Three conjectures

In this section, we explore one of the motivations for the study of tropical oriented matroids: their connection to triangulations of products of simplices. Realizable tropical oriented matroids (*i.e.* tropical hyperplane arrangements) have a canonical bijection to regular subdivisions of products of simplices [7]. Our main conjecture is the following:

Conjecture 5.1. *There is a one-to-one correspondence between the subdivisions of the product of simplices $\Delta_{n-1} \times \Delta_{d-1}$ and the tropical oriented matroids with parameters (n, d).*

In Section 6 we will prove the backward direction, and the forward direction for triangulations of $\Delta_{n-1} \times \Delta_2$. The only missing piece of the conjecture is proving that subdivisions of $\Delta_{n-1} \times \Delta_{d-1}$ satisfy the elimination axiom.

The conjectural correspondence is as follows: Give the vertices of $\Delta_{n-1} \times \Delta_{d-1}$ the labels (i, j) for $1 \leq i \leq n$ and $1 \leq j \leq d$. Given a triangulation T of $\Delta_{n-1} \times \Delta_{d-1}$, we define the **type** of a face F of T to equal $(S_1, \ldots, S_n)$, where S_i consists of those j for which (i, j) is a vertex of F. Consider the types of the faces which contain at least one vertex from each of the n copies of Δ_{d-1}; *i.e.*, those whose types have no empty coordinates. We conjecture that this is the collection of types of a tropical oriented matroid and, conversely, that every tropical oriented matroid arises in this way from a unique subdivision.

Consider, for example, the triangulation of the prism $\Delta_1 \times \Delta_2 = 12 \times 123$ shown in the left panel of Figure 5; it consists of the three tetrahedra $\{(1,1),(1,2),(1,3),(2,1)\}$, $\{(1,2),(1,3),(2,1),(2,3)\}$ and $\{(1,2),(2,1),(2,2),(2,3)\}$. These tetrahedra have types $(123, 1), (23, 13)$, and $(2, 123)$, and these are the vertices of a tropical oriented matroid with parameters $(2, 3)$. Higher dimensional types of the tropical oriented matroid correspond to lower dimensional faces of the triangulation.

A proof of Conjecture 5.1 would give us a notion of duality for tropical oriented matroids. This duality exists for tropical hyperplane arrangements [7], since regular subdivisions of $\Delta_{n-1} \times \Delta_{d-1} \cong \Delta_{d-1} \times \Delta_{n-1}$ are in canonical bijection with (n, d)-hyperplane arrangements and (d, n)-hyperplane arrangements. One can then guess how this should extend to tropical oriented matroids in general.

Definition 5.2. *A* ***semitype*** *(with parameters (n, d)) is given by an n-tuple of subsets of $[d]$, not necessarily nonempty. Given a tropical oriented matroid M, its* ***completion*** *$\widetilde{M}$ consists of all semitypes which result from types of M by changing some subset of the coordinates to the empty set. Given a collection of semitypes, its* ***reduction*** *consists of all honest types contained in the collection.*

Definition 5.3. *Let A be a semitype with parameters (n, d). Then the* ***transpose*** *A^T of A, a semitype with parameters (d, n) (*i.e.* a d-tuple of subsets of $[n]$), has $i \in A^T_j$ whenever $j \in A_i$.*

Essentially, a type can be thought of as a 0-1 $n \times d$ matrix (or alternatively a bipartite graph with n left vertices and d right vertices), which can be interpreted as either an n-tuple of subsets of $[d]$, or a d-tuple of subsets of $[n]$. The transpose operation is simply the obvious map between the two.

Definition 5.4. *Let M be a tropical oriented matroid. Then the* ***dual*** *of M is the reduction of the collection of semitypes given by transposes of semitypes in $\widetilde{M}$.*

In other words, the dual is just the reinterpretation of the types of M as d-tuples of subsets of $[n]$ instead of the other way around; the reduction and completion operations are required for purely technical reasons and are inessential to the intuition. By definition, if the dual M^* of M is indeed a tropical matroid, then clearly $M^{**} = M$.

As previously mentioned the dual of a realizable tropical oriented matroid is again a realizable tropical oriented matroid, and Conjecture 5.1 would show that this operation works in general.

Conjecture 5.5. *The dual of a tropical oriented matroid with parameters* (n, d) *is a tropical oriented matroid with parameters* (d, n)*.*

Note that as in ordinary matroid theory, the operations of deletion and contraction are dual to each other.

One should also be able to prove a topological representation theorem for tropical oriented matroids, as follows.

Definition 5.6. *A* ***tropical pseudohyperplane*** *is a subset of* $\mathbb{TP}^{d-1}$ *which is PL-homeomorphic to a tropical hyperplane.*

Conjecture 5.7. (Topological representation theorem.) *Every tropical oriented matroid can be realized by an arrangement of tropical pseudohyperplanes.*

Let us sketch the idea of a proof of Conjecture 5.7. The first step is to apply the Cayley trick to biject a triangulation of a product of simplices $\Delta_{n-1} \times \Delta_{d-1}$ to a mixed subdivision of the dilated simplex $n\Delta_{d-1}$.

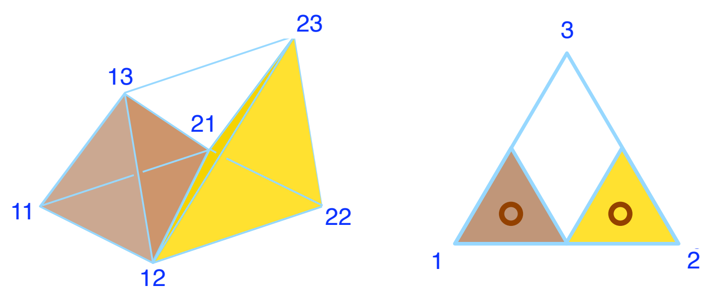

FIGURE 5.
The Cayley trick.

This process is detailed in [17]; it is a standard trick in polyhedral geometry. Let us briefly illustrate this correspondence for the tiling of $\Delta_1 \times \Delta_2$ shown in Figure 5: the tetrahedra have types $(123, 1), (23, 13)$, and $(2, 123)$, so they get mapped to the Minkowski sums $123 + 1$, $23 + 13$, and $2 + 123$. These pieces form a mixed subdivision of the triangle $2\Delta_2$, as shown.

The second step is to consider the **combinatorial dual** of this mixed subdivision of $n\Delta_{d-1}$. One possible geometric realization of this dual is the **mixed Voronoi subdivision**, defined as follows. The **Voronoi subdivision** of a k-simplex divides it into k regions, where region i consists of the points in the simplex for which i is the closest vertex. We subdivide each cell $S_1 + S_2 + \cdots + S_n$ in our mixed subdivision into the regions of the form $R_1 + R_2 + \cdots + R_n$, where R_i is a region in the Voronoi subdivision of S_i. For example, in two dimensions the finest mixed cells we can get are a triangle and a rhombus, and their mixed Voronoi subdivisions are shown in Figure 9.

The lower-dimensional faces introduced by this mixed Voronoi subdivision will all fit together to form a tropical pseudohyperplane arrangement; each simplex in the mixed subdivision corresponds to one apex of a tropical pseudohyperplane, and the other cells dictate how these pseudohyperplanes propagate throughout the diagram. This process is shown in Figure 6 for a mixed subdivision of $4\Delta_2$.[3] To recover the collection of types, since each of the n tropical pseudohyperplanes divides the figure into d canonically indexed sectors, one can simply take the types of all regions (of all dimensions) in the tropical pseudohyperplane arrangement. These are precisely the types of the triangulation we started with.

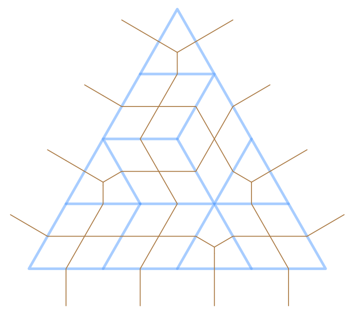

FIGURE 6.
How to obtain a tropical pseudohyperplane arrangement from a Cayley trick picture of a triangulation (here of $\Delta_3 \times \Delta_2$.)

For an example in three dimensions, the top panel of Figure 7 shows a mixed subdivision of $2\Delta_3$ (which corresponds to a triangulation of $\Delta_1 \times \Delta_3$) and the Voronoi mixed subdivision of each one of its four cells. The bottom panel shows how the lower-dimensional faces introduced by this mixed Voronoi subdivision fit together to form two tropical pseudohyperplanes, shown in different colors. As one should expect, these two pseudohyperplanes intersect in a tropical pseudoline, which is dotted in the diagram.

We conclude this section with three possible applications of these ideas.

Firstly, one may be able to use the topological representation theorem for tropical oriented matroids to prove that the triangulations of $\Delta_{n-1} \times \Delta_{d-1}$ are flip-connected. It is a natural question to ask whether the triangulations of a polytope P can all be reached from one another by a series of certain local moves, known as flips. This is not true in general [18], and there are only a few polytopes whose triangulations are known to be flip-connected, including convex polygons, cyclic polytopes [15], and products $\Delta_{n-1} \times \Delta_2$ [17]. In contrast, the analogous statement is true for the regular triangulations of an arbitrary convex polytope P. This is perhaps not surprising: as we continuously (and generically) navigate the space of height functions on the vertices of P, one expects that the corresponding regular triangulations will change in a local, predictable way. Assuming Conjecture 5.7, tropical pseudohyperplane arrangements constitute a continuous model for the triangulations of $\Delta_{n-1} \times \Delta_{d-1}$, and moving continuously around the parameter space of tropical pseudohyperplane arrangements may give a proof of the flip connectivity of these triangulations.

Secondly, tropical oriented matroids may give a proof of a conjecture describing the possible locations of the simplices in a fine mixed subdivision of $n\Delta_{d-1}$. In studying the Schubert calculus

[3]Note that tropical lines are usually drawn with angles of $90°, 135°, 135°$, while here they appear, more symmetrically, with three angles of $120°$.

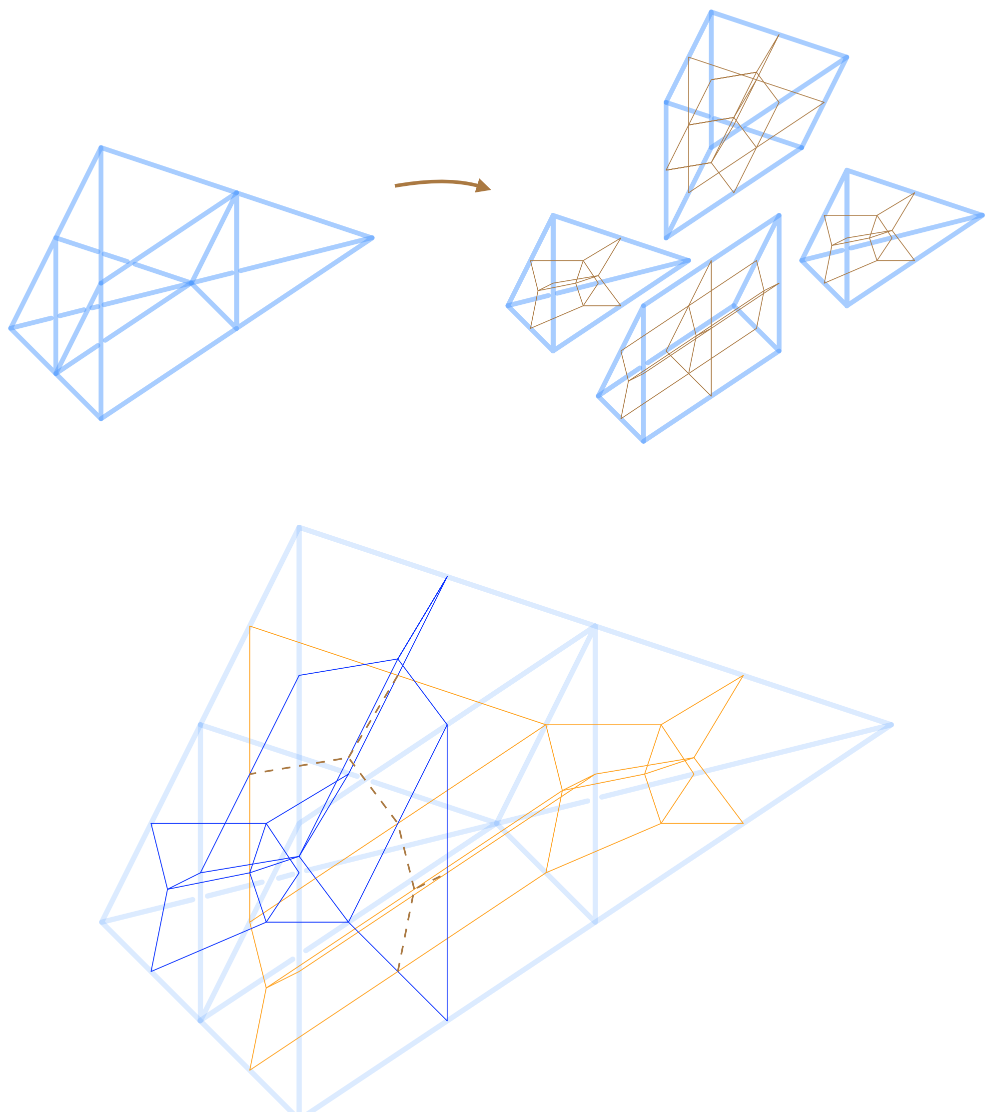

Figure 7.
A mixed subdivision of $2\Delta_3$, the Voronoi subdivision of each cell, and the resulting tropical pseudohyperplane arrangement in $\mathbb{TP}^3$.

of the flag manifold, the first author and Sara Billey [1] described the matroid $\mathcal{T}_{n,d}$ of the line arrangement determined by intersecting d generic flags in $\mathbb{R}^n$. They showed that this matroid is closely related to the fine mixed subdivisions of $n\Delta_{d-1}$: every such subdivision has exactly n simplices, which are a basis of the matroid $\mathcal{T}_{n,d}$. In the converse direction, they conjectured that every basis comes from such a subdivision, and they proved it for $d = 3$. To prove this statement, one would need a good way of constructing subdivisions. This may be approached by taking advantage of the continuous model of tropical pseudohyperplane arrangements, or by developing a toolkit for building tropical oriented matroids, in analogy with the multiple constructions available in ordinary matroid theory.

Thirdly, if Conjecture 5.1 is true, it will follow [17, 23] that there are non-realizable tropical oriented matroids of type (n, d) if and only if there are (ordinary) oriented matroids of $n+d$ elements and rank d which are not realizable over the reals. It would be very interesting to explain this connection further, and to explore the realizability of tropical oriented matroids as a tool to study the realizability of ordinary oriented matroids.

6. Tropical oriented matroids and subdivisions of $\Delta_{n-1} \times \Delta_{d-1}$.

In this section we provide strong evidence for Conjecture 5.1, which states that tropical oriented matroids are equivalent to subdivisions of $\Delta_{n-1} \times \Delta_{d-1}$. We prove one direction of the conjecture for all n and d, and the other direction in the special case of triangulations of $\Delta_{n-1} \times \Delta_2$.

To do so, we first give a combinatorial characterization of these subdivisions. Each vertex of $\Delta_{n-1} \times \Delta_{d-1}$ corresponds to an edge of the bipartite graph $K_{n,d}$. We will label the "left" vertices $\{1, \ldots, n\}$ and the "right" vertices $\{1, \ldots, d\}$, and the edge from the left vertex i to the right vertex j will be called ij. The vertices of each subpolytope in $\Delta_{n-1} \times \Delta_{d-1}$ determine a subgraph of $K_{n,d}$. Each subdivision of $\Delta_{n-1} \times \Delta_{d-1}$ is then encoded by a collection of subgraphs of $K_{n,d}$ corresponding to the full-dimensional cells of the subdivision. Figure 8 shows the three trees that encode the triangulation of Figure 5.

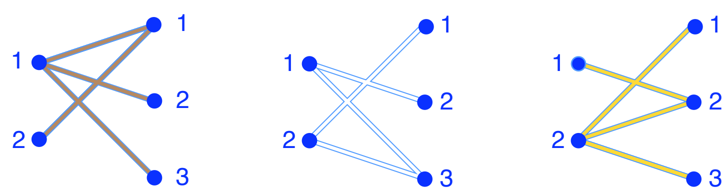

Figure 8.
The trees corresponding to the triangulation of Figure 5.

One can easily check that a subgraph t of $K_{n,d}$ encodes a full-dimensional polytope P_t if and only if t spans $K_{n,d}$. A **cut** of t is a minimal set of edges whose removal disconnects t (necessarily into two connected components C_1 and C_2). A cut is **unmixed** if it does not contain an edge joining the left part of C_1 with the right part of C_2 **and** an edge joining the right part of C_1 with the left part of C_2. Say a subgraph of t is a **facet** if it is obtained from t by removing an unmixed cut.

Lemma 6.1. *Let t be a spanning subgraph of $K_{n,d}$. The facets of the polytope P_t are precisely the polytopes P_s where s is a facet of t.*

Proof. A facet P_s of P_t is maximal with respect to some direction $(v_1, \ldots, v_n, w_1, \ldots, w_d) \in \mathbb{R}^{n+d}$, and corresponds to the subgraph s of t consisting of the edges ij with maximum weight $v_i + w_j$. Since P_s has codimension 1, s has two components C_1 and C_2. Assume that C_1 contains the left vertices $1, \ldots, k$ and the right vertices $1, \ldots, l$ and C_2 contains the left vertices $k+1, \ldots, n$ and the right vertices $l+1, \ldots, d$. Since $v_a + w_b$ takes the same value for all edges ab in s, it follows that $v_1 = \cdots = v_k = v$, $v_{k+1} = \cdots = v_n = V$, $w_1 = \cdots = w_l = w$, and $w_{l+1} = \cdots = w_d = W$ where $v + w = V + W$. Therefore s contains all the edges of t contained within C_1 or C_2, and is obtained by removing the cut which splits t into C_1 and C_2. Now we show that this cut is unmixed: if $t - s$ contains an edge ij with $1 \leq i \leq k$ and $l+1 \leq j \leq d$, that edge has weight $v + W < v + w$; so $W < w$. If $t - s$ also contains an edge $i'j'$ with $k+1 \leq i' \leq n$ and $1 \leq j' \leq l$, that will imply

$W > w$, a contradiction. It follows that s is a facet of t. Conversely, if s is a facet of t, the previous argument gives us a direction with respect to which P_s is the maximum facet of P_t. □

We then have the following combinatorial characterization of subdivisions of $\Delta_{n-1} \times \Delta_{d-1}$. The proof of this statement in the case of triangulations, which was given in [1], extends immediately to this more general setting.

Theorem 6.2. [1, 20] *A collection of subgraphs $t_1, \ldots, t_k$ of $K_{n,d}$ encodes a subdivision of $\Delta_{n-1} \times \Delta_{d-1}$ if and only if:*

(1) *Each t_i spans $K_{n,d}$.*
(2) *For each t_i and each facet s_i of t_i, either s_i has an isolated vertex or there is another t_j containing s_i.*
(3) *If there are two subgraphs t_i and t_j and a cycle C of $K_{n,d}$ which alternates between edges of t_i and edges of t_j, then both t_i and t_j contain C.*

For triangulations, in (1) we need each t_i to be a spanning tree, in (2) the subgraph s_i can be $t_i - e$ for any edge e of t_i, and in (3) one cannot have a cycle alternating between t_i and t_j.

Intuitively, (1) guarantees that the pieces of the subdivision are full-dimensional; (2) says that if we walk out of one of the pieces through one of its facets, we will either walk out of the polytope or into another piece of the subdivision; (3) guarantees that the pieces intersect face-to-face.

Theorem 6.3. *The types of the vertices of a tropical oriented matroid M with parameters (n, d) describe a subdivision of $\Delta_{n-1} \times \Delta_{d-1}$.*

Proof. Any (n, d)-type $A = (A_1, \ldots, A_n)$ with $A_i \subseteq [d]$ determines a subgraph t_A of $K_{n,d}$ as follows: the edge ij is in t_A if and only if $j \in A_i$. Notice that this graph is related to the subgraph G_A of $[d]$ of Definition 4.1 as follows: vertices i and j are connected in the graph G_A of a type A if and only if, in the bipartite graph t_A, there is some vertex in $[n]$ connected by an edge to both i and j in $[d]$.

Now consider the collection of subgraphs t_A of $K_{n,d}$ corresponding to the types A of the vertices of M; we proceed to check the conditions of Theorem 6.2.

(1) Let A be the type of any vertex of M. Since G_A is connected, the right vertices $1, \ldots, d$ are in the same connected component of t_A. Since A has no empty positions, every left vertex in $\{1, \ldots, n\}$ is connected to at least one right vertex in t_A. Therefore t_A is spanning.

(2) Let $t_i = t_A$ and $s = t_B$ correspond to types A and B and assume s has no isolated vertices. For example, consider $A = (127, 4689, 2567, 135, 34, 78, 79, 7)$ and $B = (12, 46, 256, 135, 34, 78, 79, 7)$. Each one of the graphs t_B and G_B has two connected components. Say the components of G_B are $S_1, S_2 \subseteq [d]$; in this case they are $\{1, \ldots, 6\}$ and $\{7, 8, 9\}$. By the definition of facet, B is the refinement of A with respect to one of the ordered partitions (S_1, S_2) or (S_2, S_1) of $[d]$, and is therefore a type in M. Now we need to produce a second vertex of M containing B.

This is reminiscent of Theorem 4.6, and in fact we are in Case 2 of its proof. We follow it in our example where $B = (12, 46, 256, 135, 34, 78, 79, 7)$. That proof constructs a type $C = (12, 46, 256, 135, 34, *, *, *)$ containing B, which is identical to B in the coordinates contributing to the first component $[6]$ of G_B, and adds some elements of $[6]^*$ to the coordinates corresponding to the second component $\{7, 8, 9\}$. This makes G_C connected, so C is a vertex. In the same way, we could have constructed a vertex C' containing B of the form $C' = (*, *, *, *, *, 78, 79, 7)$. These types C and C' are distinct, so one of them must not be A, and it will give us the second subgraph t_j containing s.

(3) A cycle C of $K_{n,d}$ which alternates between edges of t_A and t_B gives rise to a cycle in the comparability graph $CG(A, B)$ involving the right vertices of C. That cycle must be undirected, so C must be contained in both t_A and t_B. □

Given a (possibly nonregular) subdivision S of $\Delta_{n-1} \times \Delta_{d-1}$, define **the (n, d)-types of S** to be the (n, d)-types corresponding to the cells which contain at least one element from each of the n copies of Δ_{d-1}.

Proposition 6.4. *The (n, d)-types of any subdivision of $\Delta_{n-1} \times \Delta_{d-1}$ satisfy the boundary, comparability, and surrounding axioms of a tropical oriented matroid.*

Proof. The boundary axiom is easy, since for each $i \in [d]$, $\{(1, i), \ldots, (n, i)\}$ is a face of $\Delta_{n-1} \times \Delta_{d-1}$: it is one of the d copies of Δ_{n-1}. If the comparability graph $CG_{A,B}$ of two types A and B contained a directed cycle, then their corresponding bipartite graphs t_A and t_B would overlap on a cycle which they don't both contain. Finally, suppose A is a type corresponding to the polytope C and $P = (P_1, \ldots, P_r)$ is an ordered partition of $[d]$. We need to prove that the polytope C_P whose type is A_P is a cell of the subdivision. We do it by constructing a vector $v = (v_1, \ldots, v_n, w_1, \ldots, w_d) \in \mathbb{R}^{n+d}$ as follows: for $1 \leq j \leq d$, w_j equals the unique index k such that $j \in P_k$; and for $1 \leq i \leq n$, $v_i = -m(i)$ where $m(i)$ is the largest index such that $A_i \cap P_{m(i)} \neq \emptyset$. Then C_P is the maximal face of C with respect to v, and is therefore a cell in the subdivision. □

Theorem 6.5. *The $(n, 3)$ types of any triangulation of $\Delta_{n-1} \times \Delta_2$ form a tropical oriented matroid.*

Proof. By Proposition 6.4, it remains to check the elimination axiom. We saw that, for tropical hyperplane arrangements, eliminating between types A and B at position i amounts to walking along a tropical line from A to B, and finding its largest intersection with hyperplane i. To mimic that proof, we need to understand how to walk around the triangulation.

One can understand this in at least two ways: using the Voronoi picture or Theorem 6.2. Let us first describe how to walk around the mixed subdivision corresponding to the given triangulation. The possible puzzle pieces in a mixed subdivision of the Minkowski sum of $n\Delta_2$ are $A = 123$, $B = 13 + 23$, $C = 12 + 23$, and $D = 12 + 13$, as shown in Figure 9.

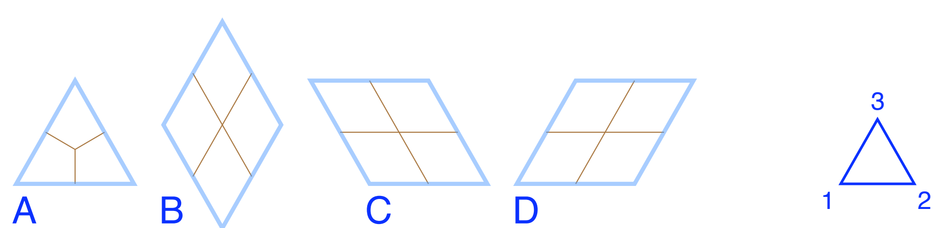

FIGURE 9.
The puzzle pieces in a Voronoi picture corresponding to a mixed subdivision, and the triangle Δ_2.

Armed with these Voronoi cells, we can say in great detail what the neighbors of a given vertex look like. For instance, consider a vertex coming from a triangle (a type A vertex.) This vertex has type $(123, *, \ldots, *)$, where each $*$ is a singleton. If we exit the triangle on the bottom side along the tropical hyperplane, then we either encounter the 12-edge of $n\Delta_d$ (if and only if each $*$ is 1 or 2), or we run into puzzle piece C or D. If the adjacent piece is C, then following the ray we are in, some index 3 in some other coordinate will be changed to 23 when we reach the vertex at the center of the puzzle piece. Similarly, if puzzle piece D is adjacent to our A piece along the 12-edge, then some index 3 will be changed to 13. So we have the following transition rule:

$$(12\mathbf{3}, 3) \to (12, 3x);$$

i.e., whenever we have a type with 123 in it, and a 3 anywhere else, then we can find an adjacent vertex which loses the boldface **3** and changes some 3 to either 31 or 32. (Note that this is only the case for one 3-singleton coordinate, and we do not know which one.)

Let us now describe the same transition rule from the point of view of Theorem 6.2. If we are in a cell C_i described by tree t_i and wish to walk to a neighboring cell, we first choose the facet of C_i that we wish to cross, which is described by the forest $t_i - e$ for some edge e of t_i. If edge e is a leaf of t_i, that facet is on the boundary of the triangle; if we cross it, we will walk out of the triangle. If, instead, e is an internal edge, then we will cross to some cell C_j which must be described by a tree of the form $(t_i - e) \cup f$. The edge f must join the two connected components of $t_i - e$. This is precisely what condition (3) of Theorem 6.2 describes.

Now imagine that we have a triangle with type $(123, *, \ldots, *)$ where each $*$ is a singleton, and we wish to exit the triangle along the bottom edge. This amounts to removing the 3 from the first coordinate. In the corresponding tree, that disconnects vertex 3 from vertices 1 and 2. To reconnect them, one must either

o add an edge from another vertex, which is already connected to 3, to vertex 1 or 2, or

o add an edge from another vertex, which is already connected to 1 or 2, to vertex 3.

However the second possibility is not an option, since it would cause our two trees to overlap on a cycle. So we must follow the first possibility, which once again is encoded as:

$$(12\mathbf{3}, 3) \to (12, 3x).$$

We can use similar logic to obtain all of the transition rules. Here are the rules for type A and type C vertices (type B and D vertices are isomorphic to type C vertices upon permutation of the numerals 1, 2, 3):

$$\begin{aligned}
(\mathbf{1}23, 1) &\to (23, 1x) \\
(1\mathbf{2}3, 2) &\to (13, 2x) \\
(12\mathbf{3}, 3) &\to (12, 3x)
\end{aligned}$$

$$\begin{aligned}
(\mathbf{1}2, 23, 1) &\to (2, 23, 1x) \\
(1\mathbf{2}, 23, x) &\to (1, 23, x1) \text{ or } (1, 123, x) \\
(12, \mathbf{2}3, x) &\to (12, 3, x3) \text{ or } (123, 3, x) \\
(12, 2\mathbf{3}, 3) &\to (12, 2, 3x)
\end{aligned}$$

So for a triangle, the generalized rule is that we remove some numeral from 123 and then add something to another appearance of that numeral (intuitively, "moving away from 1" to change $(\mathbf{1}23, 1)$ to $(23, 1x)$). For a rhombus, we either remove a non-duplicated numeral and add something to another appearance of that numeral ("moving away from 3" to change $(12, 2\mathbf{3}, 3)$ to $(12, 2, 3x)$), or remove a duplicated numeral and add the resulting singleton somewhere else ("moving towards 3" to change $(12, \mathbf{2}3, x)$ to $(12, 3, x3)$ or $(123, 3, x)$.) All of this agrees with the intuition from ordinary tropical hyperplane arrangements, and just corresponds to changing the actual coordinates in the manner indicated (or, in the regular triangulations picture, modifying the coordinates of the face-defining hyperplane.)

With these transition rules, which encode the ways one can move around a triangulation of $\Delta_{n-1} \times \Delta_2$, we are ready to prove the elimination axiom. Given two types $A = (A_1, \ldots, A_n)$ and $B = (B_1, \ldots, B_n)$, and a position i, we need to find a type C such that $C_i = A_i \cup B_i$ and C_j is either A_j, B_j, or $A_j \cup B_j$ for every other $j \neq i$. Since we have a triangulation, any subset of a type is a type, which means that we can reduce to the situation where both A and B are maximal among the purported collection of types (*i.e.* are full-dimensional cells of the triangulation.) All such types either have one triplet 123 and all singletons otherwise, or have two non-identical duplets and all singletons otherwise.

Our proof of this is by induction on the number of positions j where A_j and B_j do not satisfy a containment relation; call this number the non-containment index of the pair (A, B). Obviously if this number is zero, we can take whichever of A and B has larger i-th coordinate, and this will satisfy the requirements of the elimination axiom (since it is *a fortiori* equal to either A or B on all other coordinates.) If this number is not zero, we will modify either A or B slightly to descend. In other words, suppose we modify A; we will produce a type A' for which A' and B satisfy more containment relations than A' and B, and for which each of $\{A'_j, B_j, A'_j \cup B_j\}$ contains one of $\{A_j, B_j, A_j \cup B_j\}$ for every j, with $A'_i \cup B_i \supseteq A_i \cup B_i$. We then eliminate between A' and B by the inductive hypothesis, and if necessary remove extra elements to satisfy the elimination axiom between A and B.

We will carry out this plan by a case-by-case analysis of the possibilities for A and B. For each case, we will describe the types A and B in the top and bottom rows of a matrix, listing the coordinates of both types in the positions where either has a non-singleton; there are an arbitrary number of other coordinates j, each of which has both A_j and B_j as a singleton. We will list the possible pairs for these singletons as well; all other pairs will be ruled out due to incomparability. In each case we modify either A and B by using the transition rules to remove one numeral, indicated in boldface, from one of the coordinates.

The following cases will cover all possibilities up to isomorphism. We will explain every case that involves a new idea; every other case is essentially identical to one of the ones preceding it.

Case 1. There is some position in which both types are duplets. In this case, it is easy to check that there is exactly one such position.

Case 1a. $\begin{pmatrix} 12 & 23 & 2 \\ 12 & 3 & 2\mathbf{3} \end{pmatrix}$ Possible singletons: $\begin{pmatrix} 1 & 2 & 3 & 1 & 2 \\ 1 & 2 & 3 & 3 & 3 \end{pmatrix}$

If $i = 3$, then we are already done. Otherwise, we can apply the rule to B sending $(12, 2\mathbf{3}, 3)$ to $(12, 2, 3x)$. This gets rid of the boldface $\mathbf{3}$, and adds x to some other singleton 3; the resulting type B' is unchanged in every other position. If this is in position 2, it must end up as 23 (otherwise B' is incomparable with A), whereupon we must actually have produced A and are done. Otherwise, the only possible cases are that it sends some other 3 in position $j > 3$ to 13 (whereupon by comparing with A, the corresponding position of A must have a 1 in it) or that it sends 3 to 23 (similarly, the corresponding position of A must have a 2 in it in this case.) In either of these cases, another containment relation is created, as desired by our induction.

Case 1b. $\begin{pmatrix} 12 & 2\mathbf{3} & 3 \\ 12 & 2 & 13 \end{pmatrix}$ Possible singletons: $\begin{pmatrix} 1 & 2 & 3 & 3 & 3 \\ 1 & 2 & 3 & 1 & 2 \end{pmatrix}$

Case 1c. $\begin{pmatrix} 12 & \mathbf{2}3 & 2 \\ 13 & 3 & 12 \end{pmatrix}$ Possible singletons: $\begin{pmatrix} 1 & 2 & 3 & 1 & 2 & 2 \\ 1 & 2 & 3 & 3 & 1 & 3 \end{pmatrix}$

Case 1d. $\begin{pmatrix} 12 & \mathbf{2}3 & 2 \\ 13 & 3 & 23 \end{pmatrix}$ Possible singletons: $\begin{pmatrix} 1 & 2 & 3 & 1 & 2 & 2 \\ 1 & 2 & 3 & 3 & 1 & 3 \end{pmatrix}$

Case 1e. $\begin{pmatrix} 12 & \mathbf{2}3 & 1 \\ 23 & 3 & 12 \end{pmatrix}$ Possible singletons: $\begin{pmatrix} 1 & 2 & 3 & 1 & 1 & 2 \\ 1 & 2 & 3 & 2 & 3 & 3 \end{pmatrix}$

Case 2. Both types come from rhombi, and there is no overlap among the positions in which they have duplets.

Case 2a. $\begin{pmatrix} 12 & 13 & 2 & 2 \\ 1 & 1 & \mathbf{1}2 & 13 \end{pmatrix}$ Possible singletons: $\begin{pmatrix} 1 & 2 & 3 & 2 & 2 & 3 \\ 1 & 2 & 3 & 1 & 3 & 1 \end{pmatrix}$

As before, if $i = 3$, we are done. Otherwise, we remove the boldface **1** using the rule $(\mathbf{1}2, 13, x) \to ((2, 13, x2) \text{ or } (2, 123, x))$, which changes some x to $x2$ (or 13 to 123). If this is position 1, the non-containment index remains constant, but we have reduced to case 1 and thus are done. If not, the non-containment index decreases (by the same logic as before), and we are again done.

Case 2b. $\begin{pmatrix} 12 & 13 & 2 & 2 \\ 1 & 3 & \mathbf{1}2 & 13 \end{pmatrix}$ Possible singletons: $\begin{pmatrix} 1 & 2 & 3 & 1 & 2 & 2 \\ 1 & 2 & 3 & 3 & 1 & 3 \end{pmatrix}$

Case 2c. $\begin{pmatrix} 12 & 13 & 2 & 3 \\ 1 & 1 & 12 & 13 \end{pmatrix}$ Possible singletons: $\begin{pmatrix} 1 & 2 & 3 & 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 2 & 3 & 3 & 2 \end{pmatrix}$

At this point we are not ready to deal with case 2c; we revisit it after case 2m.

Case 2d. $\begin{pmatrix} 12 & 13 & 3 & 3 \\ 1 & 1 & 12 & \mathbf{2}3 \end{pmatrix}$ Possible singletons: $\begin{pmatrix} 1 & 2 & 3 & 2 & 3 & 3 \\ 1 & 2 & 3 & 1 & 1 & 2 \end{pmatrix}$

Case 2e. $\begin{pmatrix} 12 & 13 & 2 & 2 \\ 1 & 1 & \mathbf{1}2 & 23 \end{pmatrix}$ Possible singletons: $\begin{pmatrix} 1 & 2 & 3 & 2 & 2 & 3 \\ 1 & 2 & 3 & 1 & 3 & 1 \end{pmatrix}$

Eliminating the boldface **1** changes some 1 to $1x$. If it changes position 1 to 12 or position 2 to 13, we reduce to case 1. It can't change position 1 to 13 or position 2 to 12 because of incomparability. The only other case where the non-containment index does not decrease is if it changes $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ to $\begin{pmatrix} 2 \\ 13 \end{pmatrix}$, but this is a reduction to a situation isomorphic to case 2d.

Case 2f. $\begin{pmatrix} 12 & 13 & 2 & 3 \\ 1 & 1 & \mathbf{1}2 & 23 \end{pmatrix}$ Possible singletons: $\begin{pmatrix} 1 & 2 & 3 & 2 & 3 & 3 \\ 1 & 2 & 3 & 1 & 1 & 2 \end{pmatrix}$

Case 2g. $\begin{pmatrix} 12 & 13 & 1 & 3 \\ 2 & 1 & 12 & \mathbf{2}3 \end{pmatrix}$ Possible singletons: $\begin{pmatrix} 1 & 2 & 3 & 1 & 3 & 3 \\ 1 & 2 & 3 & 2 & 1 & 2 \end{pmatrix}$

Case 2h. $\begin{pmatrix} 12 & 13 & 3 & 3 \\ 2 & 1 & 12 & \mathbf{2}3 \end{pmatrix}$ Possible singletons: $\begin{pmatrix} 1 & 2 & 3 & 1 & 3 & 3 \\ 1 & 2 & 3 & 2 & 1 & 2 \end{pmatrix}$

Case 2i. $\begin{pmatrix} \mathbf{1}2 & 13 & 1 & 1 \\ 2 & 2 & 12 & 23 \end{pmatrix}$ Possible singletons: $\begin{pmatrix} 1 & 2 & 3 & 1 & 1 & 3 \\ 1 & 2 & 3 & 2 & 3 & 2 \end{pmatrix}$

Case 2j. $\begin{pmatrix} \mathbf{1}2 & 13 & 1 & 3 \\ 2 & 2 & 12 & 23 \end{pmatrix}$ Possible singletons: $\begin{pmatrix} 1 & 2 & 3 & 1 & 1 & 3 & 3 \\ 1 & 2 & 3 & 2 & 3 & 1 & 2 \end{pmatrix}$

At most one of $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ can occur, but this does not affect the argument: removing the boldface **1** adds a 2 somewhere, which must reduce the non-containment index.

Case 2k. $\begin{pmatrix} \mathbf{1}2 & 13 & 3 & 3 \\ 2 & 2 & 12 & 23 \end{pmatrix}$ Possible singletons: $\begin{pmatrix} 1 & 2 & 3 & 1 & 3 & 3 \\ 1 & 2 & 3 & 2 & 1 & 2 \end{pmatrix}$

Case 2l. $\begin{pmatrix} 12 & 13 & 1 & 3 \\ 2 & 3 & 1\mathbf{2} & 23 \end{pmatrix}$ Possible singletons: $\begin{pmatrix} 1 & 2 & 3 & 1 & 1 & 3 \\ 1 & 2 & 3 & 2 & 3 & 2 \end{pmatrix}$

If the boldface **2**'s removal produced 123 in position 4, we would have a problem, but this is not possible due to comparability.

Now we are ready to deal with case 2c.

Case 2c. $\begin{pmatrix} 1\mathbf{2} & 13 & 2 & 3 \\ 1 & 1 & 12 & 13 \end{pmatrix}$ Possible singletons: $\begin{pmatrix} 1 & 2 & 3 & 2 & 3 & 2 & 3 \\ 1 & 2 & 3 & 1 & 1 & 3 & 2 \end{pmatrix}$

Removing the boldface **2** changes some 2 to $2x$. If this is in position 3, we are done by reduction to case 1. Otherwise, the only case in which this does not produce an extra containment relation is when we change $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ to $\begin{pmatrix} 23 \\ 2 \end{pmatrix}$ or $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ to $\begin{pmatrix} 23 \\ 1 \end{pmatrix}$. The results are isomorphic to cases 2l and 2j, respectively, so we are done.

Case 3. At least one of the types comes from a triangle (type A) vertex.

We can assume that the matrix contains $\begin{pmatrix} \mathbf{1}23 & 1 \\ \alpha & \beta \end{pmatrix}$, where α and β are subsets of $\{1, 2, 3\}$ and β is not a singleton. If $i = 1$ we are done. Otherwise remove the boldface **1** from A, changing some 1 to $1x$. Notice that α cannot contain 1 by comparability, so the type A' obtained is such that each set in $\{A'_j, B_j, A'_j \cup B_j\}$ contains one of the sets in $\{A_j, B_j, A_j \cup B_j\}$ (for all $j \neq i$), and $A'_i \cup B_i \supseteq A_i \cup B_i$. This transformation might not reduce the non-containment index, but it makes the triangle A into a rhombus A'. If B is also a triangle, we can make it a rhombus in the same way, and then invoke cases 1 and 2.

This completes the typology and thus the proof. □

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