

TREE METRICS AND LOG-CONCAVITY FOR MATROIDS

FEDERICO ARDILA–MANTILLA, SERGIO CRISTANCHO, GRAHAM DENHAM,
CHRISTOPHER EUR, JUNE HUH, AND BOTONG WANG

ABSTRACT. We show that a set function ν satisfies the gross substitutes property if and only if its homogeneous generating polynomial $Z_{q,\nu}$ is a Lorentzian polynomial for all positive $q \leq 1$, answering a question of Eur–Huh. We achieve this by giving a rank 1 upper bound for the distance matrix of an ultrametric tree, refining a classical result of Graham–Pollak. This characterization enables us to resolve open problems that strengthen Mason’s log-concavity conjectures for the numbers of independent sets of a matroid: one posed by Giansiracusa–Rincón–Schleis–Ulirsch for valuated matroids, and two posed by Dowling in 1980 and Zhao in 1985 for ordinary matroids.

1. INTRODUCTION

Let E be a finite set of cardinality n . A *matroid* M on E is given by a nonempty collection $\text{IN}(M)$ of subsets of E , called *independent sets* of M , satisfying the following properties:

- (1) If $I \in \text{IN}(M)$ and J is a subset of I , then $J \in \text{IN}(M)$.
- (2) If $I, J \in \text{IN}(M)$ and $|I| > |J|$, then there is $i \in I \setminus J$ such that $J \cup i \in \text{IN}(M)$.

The *rank* of M is the common cardinality of the maximal independent sets. For background and any undefined terminology in matroid theory, we refer to [Oxl11].

Let $I_k = I_k(M)$ be the number of independent sets of M of cardinality k . Mason [Mas72] conjectured that the following three families of inequalities hold for all $0 < k < n$:

$$\begin{aligned} \text{(M1)} \quad & I_k^2 \geq I_{k-1}I_{k+1}, \\ \text{(M2)} \quad & I_k^2 \geq \left(1 + \frac{1}{k}\right)I_{k-1}I_{k+1}, \\ \text{(M3)} \quad & I_k^2 \geq \left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{n-k}\right)I_{k-1}I_{k+1}. \end{aligned}$$

In other words, the sequences I_k , $k!I_k$, and $I_k/\binom{n}{k}$ are, respectively, *log-concave* in k . The last property is referred to as the *ultra log-concavity* of the sequence I_0, \dots, I_n . Clearly, (M3) implies (M2) implies (M1) for a given matroid.

Conjecture (M1) was proved for realizable matroids in [Len13], by building on the work of [HK12], and for arbitrary matroids in [AHK18]. Conjecture (M2) was proved in [HSW22] using the main results of [AHK18], while Conjecture (M3) was proved independently in [ALOGV24]

and [BH20]. These inequalities motivated the development of combinatorial Hodge theory [AHK18] and the theory of Lorentzian polynomials [BH20].

In Theorem 1.4, we extend (M3) from matroids on E to M^{\natural} -concave functions on 2^E , which are precisely the set functions satisfying the *gross substitutes property* from economics. This answers a question of Giansiracusa–Rincón–Schleis–Ulirsch in [GRSU24, Question 4.1], who asked whether (M3) holds for the class of M^{\natural} -concave functions constructed from valuated matroids discussed in Example 1.3. In Theorem 1.6, we prove a multivariate polynomial inequality that refines (M2), affirmatively settling conjectures of Dowling [Dow80] and Zhao [Zha85] that were recently popularized by Pak [Pak24]. Both results follow from our main result, Theorem 1.5, characterizing M^{\natural} -concave functions in terms of Lorentzian polynomials:

A set function ν on E satisfies the gross substitutes property if and only if its homogeneous generating polynomial $Z_{q,\nu} = \sum_{S \subseteq E} q^{-\nu(S)} x^S y^{|E|-|S|}$ is a Lorentzian polynomial for all positive $q \leq 1$.

This answers a question of Eur–Huh in [EH20, Section 5.1], who asked for a characterization of set functions whose homogeneous generating polynomials are Lorentzian for all positive $q \leq 1$.¹ The key ingredient is Theorem 1.8, which gives a rank 1 upper bound for the distance matrix of an ultrametric tree. This refines the classical result of Graham–Pollak that the distance matrix of a tree has exactly one positive eigenvalue. In the rest of the introduction, we describe these results in detail.

1.1. Inequalities for M^{\natural} -concave set functions. A function $\nu: 2^E \rightarrow \mathbb{R} \cup \{-\infty\}$ is said to be M^{\natural} -concave if it satisfies the *exchange property*: For any $I_1, I_2 \subseteq E$ and $i_1 \in I_1 \setminus I_2$, we have

- (1) either $\nu(I_1) + \nu(I_2) \leq \nu(I_1 \setminus i_1) + \nu(I_2 \cup i_1)$, or
- (2) there is $i_2 \in I_2 \setminus I_1$ such that $\nu(I_1) + \nu(I_2) \leq \nu((I_1 \setminus i_1) \cup i_2) + \nu((I_2 \setminus i_2) \cup i_1)$.

The *effective domain* of such a function is the set

$$\text{dom}(\nu) := \{I \subseteq E : \nu(I) \neq -\infty\}.$$

For our purposes, without loss of generality, we may suppose that $\text{dom}(\nu)$ is nonempty.

Introduced by Murota and Shioura [MS99], M^{\natural} -concave functions are central objects in discrete convex analysis. Fujishige–Yang [FY03] and Reijnierse–van Gellekom–Potters [RvGP02] independently proved that M^{\natural} -concavity is equivalent to the *gross substitutes property* in economics, introduced by Kelso and Crawford [KJC82] two decades earlier. For a comprehensive introduction to M^{\natural} -concave functions, we refer the reader to [Mur03, Chapter 6]. The notion of M^{\natural} -concave functions generalizes several concepts in matroid theory, including independent sets, rank functions, and valuated matroids.

¹The function r from [EH20, Example 5.4] is not M^{\natural} -concave, contrary to the authors' claim: the maximum of $r(\{i, j\}) + r(\{k\})$ is achieved uniquely at $k = 2$.

Example 1.1 (Matroid independent sets). Let M be a matroid on E . The *independent set indicator function* of M , defined by

$$\nu_M(S) := \begin{cases} 0 & \text{if } S \text{ is an independent subset of } M, \\ -\infty & \text{if } S \text{ is not an independent subset of } M, \end{cases}$$

is an M^{\natural} -concave function whose effective domain is the collection of independent sets $\text{IN}(M)$.

Example 1.2 (Matroid rank functions). The *rank function* of M , defined by

$$\text{rk}_M(S) := \max \left\{ |I|, I \text{ is an independent set of } M \text{ in } S \right\},$$

is an M^{\natural} -concave function whose effective domain is 2^E . More generally, any non-negative linear combination of the rank functions of the constituent matroids in a flag matroid is a M^{\natural} -concave function [Shi12, Theorem 3].

Example 1.3 (Valuated matroid independent sets). A *valuated matroid* of rank d on E is a function $\underline{\nu}: \binom{E}{d} \rightarrow \mathbb{R} \cup \{-\infty\}$ that satisfies the *symmetric exchange property*: For any d -element subsets B_1, B_2 of E and $b_1 \in B_1 \setminus B_2$, there is $b_2 \in B_2 \setminus B_1$ such that

$$\underline{\nu}(B_1) + \underline{\nu}(B_2) \leq \underline{\nu}((B_1 \setminus b_1) \cup b_2) + \underline{\nu}((B_2 \setminus b_2) \cup b_1).$$

Valuated matroids are precisely the possible height functions in a regular subdivision of a matroid polytope into matroid polytopes. [Spe08]

Murota [Mur97] considered the M^{\natural} -concave extension ν of $\underline{\nu}$ to 2^E given by

$$\nu(S) := \max \left\{ \underline{\nu}(B), B \text{ is a } d\text{-element subset of } E \text{ containing } S \right\},$$

where the maximum of the empty set is defined to be $-\infty$. The effective domain of ν is the collection of independent sets of a matroid on E , the *underlying matroid* of $\underline{\nu}$.

For a function $\nu: 2^E \rightarrow \mathbb{R} \cup \{-\infty\}$, an integer $0 \leq k \leq n$, and a real number $0 < q \leq 1$, we set

$$I_{q,\nu;k} := \sum_{I \in \binom{E}{k}} q^{-\nu(I)},$$

where, by convention, $q^\infty = 0$. This sequence was first considered by Giansiracusa, Rincón, Schleich, and Ulirsch for valuated matroids in the context of Example 1.3, who proved in [GRSU24, Theorem A] that Murota's extension ν of a valuated matroid satisfies a generalization of (M2):

$$I_{q,\nu;k}^2 \geq \left(1 + \frac{1}{k}\right) I_{q,\nu;k-1} I_{q,\nu;k+1} \quad \text{for all } 0 < k < n \text{ and } 0 < q \leq 1.$$

They asked in [GRSU24, Question 4.1] whether ν satisfies a generalization of (M3), and they provided extensive numerical evidence in support of this. We prove their prediction in the more general setting of M^{\natural} -concave functions.

Theorem 1.4. For any M^{\natural} -concave function $\nu: 2^E \rightarrow \mathbb{R} \cup \{-\infty\}$, we have

$$I_{q,\nu;k}^2 \geq \left(1 + \frac{1}{k}\right) \left(1 + \frac{1}{n-k}\right) I_{q,\nu;k-1} I_{q,\nu;k+1} \quad \text{for all } 0 < k < n \text{ and } 0 < q \leq 1.$$

We deduce Theorem 1.4 from an analytic characterization of M^{\natural} -concave functions in terms of *Lorentzian polynomials* [BH20], whose definition we recall in Section 3.1. We define the *homogeneous generating polynomial* a function $\nu: 2^E \rightarrow \mathbb{R} \cup \{-\infty\}$ by

$$Z_{q,\nu}(x, y) := \sum_{S \subseteq E} q^{-\nu(S)} x^S y^{|E \setminus S|} \quad \text{with } x^S := \prod_{i \in S} x_i,$$

where $x = (x_i)_{i \in E}$ and y is a homogenizing variable different from x_i for $i \in E$. We view $Z_{q,\nu}$ as a homogeneous polynomial of degree n in $n + 1$ variables (x, y) with a positive parameter q .

Theorem 1.5. A function $\nu: 2^E \rightarrow \mathbb{R} \cup \{-\infty\}$ is M^{\natural} -concave if and only if its homogeneous generating polynomial $Z_{q,\nu}$ is a Lorentzian polynomial for all positive $q \leq 1$.

Theorem 1.5 answers a question of Eur–Huh in [EH20, Section 5], who proved the “if” direction of Theorem 1.5 in [EH20, Proposition 5.5]. The other direction is a new contribution, which generalizes the following known results:

- (1) When $\nu = \nu_M$ is the independent set indicator function of a matroid M in Example 1.1, Theorem 1.5 recovers the statement that the *homogeneous independent set generating polynomial*

$$I(x, y) := \sum_{I \in \text{IN}(M)} x^I y^{|E \setminus I|},$$

is a Lorentzian polynomial [BH20, Section 4.3]. See [ALOGV24, Theorem 4.1] for an equivalent statement.

- (2) When $\nu = \text{rk}_M$ is the rank function of a matroid M as in Example 1.2, Theorem 1.5 recovers the statement that the *homogeneous multivariate Tutte polynomial*

$$T_q(x, y) := \sum_{S \subseteq E} q^{-\text{rk}_M(S)} x^S y^{|E \setminus S|},$$

is a Lorentzian polynomial for all positive $q \leq 1$ in [BH20, Theorem 4.10]. The Lorentzian property of the homogeneous independent set generating polynomial follows from the identity

$$I(x, y) = \lim_{q \rightarrow 0} T_q(qx, y).$$

Theorem 1.4 follows from Theorem 1.5 by identifying the variables x_i for $i \in E$, see Section 3.2. Mason’s strongest conjecture (M3) is the special case of Theorem 1.4 when $\nu = \nu_M$.

1.2. Polynomial inequalities for matroids. Let M be a matroid on E , and let $\text{IN}_k(M)$ be the collection of k -element independent sets of M . We consider the generating polynomial

$$I_k(x) := \sum_{S \in \text{IN}(M)_k} x^S, \quad \text{with } x^S := \prod_{i \in S} x_i,$$

where $x = (x_i)_{i \in E}$. For multivariate polynomials $f, g \in \mathbb{Z}[x_i]_{i \in E}$, we write $f \succeq g$ if all the coefficients of the difference $f - g$ are nonnegative. Dowling [Dow80, Section 2] conjectured that the polynomial refinement of Mason’s conjecture (M1) holds for any matroid: that is, the inequality

$I_k(x)^2 \succeq I_{k-1}(x)I_{k+1}(x)$ holds for all k , coefficient by coefficient. Subsequently, Zhao [Zha85] made the stronger conjecture that the polynomial version of (M2) holds for all matroids.

We prove Zhao's and Dowling's conjectures. Using Theorem 1.5, we establish a stronger polynomial inequality for the more general class of M^{\natural} -concave functions. For a function $\nu : 2^E \rightarrow \mathbb{R} \cup \{-\infty\}$, we set

$$I_{q,\nu;k}(x) := \sum_{S \in \binom{E}{k}} q^{-\nu(S)} x^S.$$

When ν is the independent set indicator function of M , the following theorem states that the polynomial version of (M2), and hence also (M1), holds for M .

Theorem 1.6. For any M^{\natural} -concave function $\nu : 2^E \rightarrow \mathbb{R} \cup \{-\infty\}$, we have the coefficientwise inequality

$$I_{q,\nu;k}(x)^2 \succeq \left(1 + \frac{1}{k}\right) I_{q,\nu;k-1}(x) I_{q,\nu;k+1}(x) \text{ for all } 0 < k < n \text{ and } 0 < q \leq 1.$$

In Section 3.3, we prove the stronger² statement that, for any M^{\natural} -concave function ν and any $0 \leq i \leq j \leq k \leq l \leq n$ with $i + l = j + k$, we have

$$j! I_{q,\nu;j}(x) \cdot k! I_{q,\nu;k}(x) \succeq i! I_{q,\nu;i}(x) \cdot l! I_{q,\nu;l}(x) \text{ for all } 0 < q \leq 1.$$

For matroids, the displayed inequality admits the following combinatorial interpretation. For a matroid M on E and nonnegative integers a and b with $a + b = n$, we write $N_M(a, b)$ for the number of partitions of $E = A \sqcup B$ into ordered independent sets A and B of M of sizes a and b .

Corollary 1.7. For any matroid M on E and any $0 \leq i \leq j \leq k \leq l \leq n$ with $i + l = j + k$,

$$j! k! N_M(j, k) \geq i! l! N_M(i, l).$$

As Pak pointed out, the polynomial version of Mason's strongest inequality (M3) is not true: the sequence $I_k(x)$ need not be coefficient-wise ultra log-concave. For the uniform matroid of rank 2 on $\{1, 2\}$, we have

$$I_1(x_1, x_2)^2 / 2^2 - I_0(x_1, x_2) I_2(x_1, x_2) = \frac{1}{4}(x_1^2 + x_2^2 - 2x_1x_2) = \frac{1}{4}(x_1 - x_2)^2,$$

which is non-negative numerically, but not coefficient-wise. This illustrates the fact that Pak's polynomial refinement is significantly stronger than Mason's original conjecture.

1.3. An inequality for ultrametric trees. A central ingredient in the proof of Theorem 1.5 is a result on tree distance matrices that we now describe. In their design of efficient address systems for communication networks, Graham and Pollak [GP71] introduced the *distance matrix* of a graph, whose ij entry is the distance between vertices i and j . They showed that the signature of this matrix gives a lower bound for the addresses in their design, and proved that the distance matrix of any tree has the Lorentzian signature $(+, -, \dots, -)$. The latter fact also

²Unlike sequences of positive numbers, a sequence of positive polynomials can be locally log-concave but not globally log-concave coefficientwise; an example is the sequence $x, x + y, x + \frac{7}{4}y, 3y$.

follows from work of Schoenberg [Sch35, Theorem 1], using the fact that an ultrametric tree can be metrically embedded in an ℓ^2 -space [TV83]. Our proofs of the matroid inequalities above rely on a refinement of this result for ultrametric trees.

An *ultrametric tree* is a rooted tree with nonnegative lengths on its edges such that all the leaves are at the same distance from the root. If the tree has n leaves, let D be the $n \times n$ leaf distance matrix, whose ij entry is the distance between the leaves i and j . For $n \times n$ real symmetric matrices A and B , we write $A \geq B$ if the difference $A - B$ is positive semidefinite. We write $\mathbf{1}_{n \times n}$ for the $n \times n$ matrix all of whose entries are 1.

Theorem 1.8. For any ultrametric tree with n leaves whose common distance from the root is 1,

$$\left(1 - \frac{1}{n}\right) \mathbf{1}_{n \times n} \geq \frac{1}{2} D.$$

The constant $(1 - \frac{1}{n})$ is the smallest number that makes the above statement true.

Theorem 1.8 can be seen as a quantitative refinement of Graham and Pollak’s result that D has at most one positive eigenvalue in the ultrametric case. In Section 2.2, we characterize the ultrametric trees for which the scalar $(1 - \frac{1}{n})$ is optimal.

Theorem 1.8 serves as the base case in our inductive argument for the proof of Theorem 1.5. This connection stems from Lemma 3.3, which states that any M^{\natural} -concave function on 2^E whose effective domain contains all the subsets of E with at most 2 elements gives rise to an ultrametric tree. This is a variation of the standard fact that the space of uniform valuated matroids of rank 2 is equal to the space of phylogenetic trees [MS15, Theorem 4.3.5].

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Our results and proofs were first announced in Feb. 2025 at the Institute for Advanced Study [Ard25a, Ard25b]. In Jan. 2026, a few days after our work appeared on the arXiv, Cao, Chen, Li, and Wu [CCLW26] announced results that overlap with one of our four main theorems. We thank them for bringing Dowling and Zhao’s conjectures to our attention.

2. TREES

2.1. Ultrametric tree matrices. When T is a tree with nonnegative edge lengths, let $d(i, j)$ denote the sum of the edge lengths along the unique path in T connecting the vertices i and j . We call d the *tree distance function* for T , and we define the *diameter* of T to be $\max_{i,j} d(i, j)$.

If T is rooted, the *height* $H(i)$ of a vertex i in T is the distance from i to its furthest descendant in T . The term is chosen to match drawings of rooted trees with the root at the top. For any distinct vertices i and j in T , let $i \vee j$ denote their *lowest common ancestor*, that is, the unique vertex of T lying on all three paths connecting the root, i , and j in T .

Definition 2.1. An *ultrametric tree* T is a rooted tree with a tree distance function for which all leaves have the same distance from the root. That distance is the *radius* of the tree, which is also equal to the height of the root, and half the diameter.

If T is an ultrametric tree, the function d satisfies the *ultrametric inequality*

$$d(i, j) \leq \max \left\{ d(i, k), d(k, j) \right\} \text{ for any leaves } i, j, k.$$

It follows that the leaves of T satisfies the *three-point condition*:

The maximum among $d(i, j)$, $d(j, k)$, $d(k, i)$ is attained at least twice for any leaves i, j, k .

Every ultrametric tree with positive lengths defines an ultrametric on its leaves, and every finite ultrametric space arises in this way from an ultrametric tree [BG91, Theorem 3.1].

In this section, we prove the following extension of Theorem 1.8. An *upper subtree* U of T is an upper ideal of T when regarded as a poset, that is, a set of vertices of T such that every ancestor of a vertex in U is also in U .

Proposition 2.2. Let T be an ultrametric tree of radius 1, let U be a nonempty upper subtree of T , and let $M = U_{\min}$ be the set of minimal elements of U . For each i in M , let n_i be the number of leaves of T below i , and set $n := \sum_{i \in M} n_i$. Then the symmetric matrix $A^{T,U}$ with rows and columns indexed by M and with entries

$$A_{ij}^{T,U} = \begin{cases} (1 - \frac{1}{n}) - H(i \vee j), & \text{if } i \neq j, \\ (1 - \frac{1}{n}) - (1 - \frac{1}{n_i})H(i), & \text{if } i = j, \end{cases}$$

is positive semidefinite.

The first part of Theorem 1.8 is the special case when $U = T$.

Proof. We may assume the tree T is binary by replacing any vertex having $k > 2$ children with $k - 1$ vertices having two children each, connected by edges of length 0; this is illustrated in Figure 1.

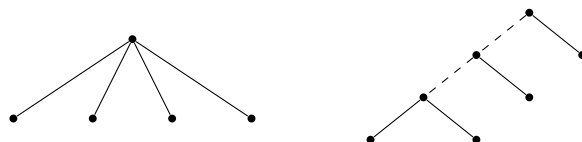


FIGURE 1. Non-binary trees can be seen as binary by adding edges of length 0.

We prove this statement for all pairs (T, U) by induction on $|V(T)| + |M|$. Notice that when $|M| = 1$, the matrix $A^{T,U}$ has a single entry which is nonnegative; this will be the base cases of our induction. When U has $|M| \geq 2$ leaves, at least one of the following statements is true:

- (1) There is a leaf i of U with no siblings in U .
- (2) There are sibling leaves i, j of U with a common parent $h = i \vee j$.

In the first case, let h and g be the parent and grandparent of i , respectively; they must both exist since $|M| \geq 2$. Let T' be the smaller tree obtained by removing h and all the descendants of h that are neither i nor descendants of i , and replacing edges (g, h) and (h, i) by an edge (g, i) of length $\ell(g, h) + \ell(h, i)$. If U' is the corresponding upper subtree of T' , then $A^{T,U} = A^{T',U'}$ and the induction hypothesis applies to (T', U') .

In the second case, let U' be the upper subtree $U \setminus \{i, j\}$ of T , whose set of minimal elements $M' = M \setminus \{i, j\} \cup \{h\}$ is smaller. We compare the matrices $A^{T,U}$ and $A^{T,U'}$. Since T is binary, h has no other children other than i and j , so

$$i \vee k = j \vee k = h \vee k \text{ for any } k \in M \setminus \{i, j\}.$$

Setting $\langle a \rangle := 1 - \frac{1}{a}$ and focusing on the i -th and j -th rows and columns of the matrix $A^{T,U}$, which are shown as the first and the second rows and columns below, we have

$$\begin{aligned}
A^{T,U} &= \begin{bmatrix} \langle n \rangle - \langle n_i \rangle H(i) & \langle n \rangle - H(h) & \langle n \rangle - H(i \vee k) & \cdots \\ \langle n \rangle - H(h) & \langle n \rangle - \langle n_j \rangle H(j) & \langle n \rangle - H(j \vee k) & \cdots \\ \langle n \rangle - H(i \vee k) & \langle n \rangle - H(j \vee k) & \langle n \rangle - \langle n_k \rangle H(k) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \text{ since } i \vee j = h \\
&\geq \begin{bmatrix} \langle n \rangle - \langle n_i \rangle H(h) & \langle n \rangle - H(h) & \langle n \rangle - H(i \vee k) & \cdots \\ \langle n \rangle - H(h) & \langle n \rangle - \langle n_j \rangle H(h) & \langle n \rangle - H(j \vee k) & \cdots \\ \langle n \rangle - H(i \vee k) & \langle n \rangle - H(j \vee k) & \langle n \rangle - \langle n_k \rangle H(k) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \text{ since } H(i), H(j) \leq H(h) \\
&\sim \begin{bmatrix} (2 - \langle n_i \rangle - \langle n_j \rangle)H(h) & (\langle n_j \rangle - 1)H(h) & 0 & \cdots \\ (\langle n_j \rangle - 1)H(h) & \langle n \rangle - \langle n_j \rangle H(h) & \langle n \rangle - H(h \vee k) & \cdots \\ 0 & \langle n \rangle - H(h \vee k) & \langle n \rangle - \langle n_k \rangle H(k) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{array}{l} \text{replacing} \\ \text{row } i \mapsto \text{row } i - \text{row } j \\ \text{col } i \mapsto \text{col } i - \text{col } j, \\ \text{using } i \vee k = j \vee k = h \vee k. \end{array} \\
&\sim \begin{bmatrix} (2 - \langle n_i \rangle - \langle n_j \rangle)H(h) & 0 & 0 & \cdots \\ 0 & \langle n \rangle - \langle n_h \rangle H(h) & \langle n \rangle - H(h \vee k) & \cdots \\ 0 & \langle n \rangle - H(h \vee k) & \langle n \rangle - \langle n_k \rangle H(k) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{array}{l} \text{replacing} \\ \text{row } j \mapsto \text{row } j + \frac{n_i}{n_h} \text{row } i \\ \text{col } j \mapsto \text{col } j + \frac{n_i}{n_h} \text{col } i. \end{array} \\
&= \left[\begin{array}{c|c} \left(\frac{1}{n_i} + \frac{1}{n_j} \right) H(h) & 0 \\ \hline 0 & A^{T,U'} \end{array} \right].
\end{aligned}$$

Therefore, $A^{T,U}$ is positive semidefinite by the induction hypothesis. The third step creates zeros in the (i, j) -th and (j, i) -th entries because

$$\frac{n_i}{n_h} = \frac{n_i}{n_i + n_j} = \frac{1 - \langle n_j \rangle}{2 - \langle n_i \rangle - \langle n_j \rangle}.$$

The formula for the (j, j) -th entry follows from the following identity for $a = n_i$ and $b = n_j$:

$$(*) \quad \langle a + b \rangle = \frac{1 - \langle a \rangle \langle b \rangle}{2 - \langle a \rangle - \langle b \rangle}.$$

This last step features a pleasant subtlety worth mentioning explicitly: our proof requires a function $\langle \cdot \rangle : \mathbb{Z}_{>0} \rightarrow \mathbb{R}_{\geq 0}$ that satisfies $(*)$. A priori, this functional equation would seem overdetermined, since it gives different formulas for $\langle a + b \rangle$ for different choices of a and b . In fact, the solutions to $(*)$ are $\langle x \rangle = 1 - \frac{1}{cx}$ for any nonzero c , and the smallest positive c such that $1 - \frac{1}{cx}$ is nonnegative for all positive x is 1. Thus, the choice of constants $\langle n_k \rangle = 1 - \frac{1}{n_k}$ in the matrix $A^{T,U}$ is exactly what ensures that the inductive machinery operates smoothly. \square

Theorem 1.8. For any ultrametric tree with n leaves whose common distance from the root is 1,

$$\left(1 - \frac{1}{n}\right) \mathbf{1}_{n \times n} \geq \frac{1}{2} D.$$

The constant $(1 - \frac{1}{n})$ is the smallest number that makes the above statement true.

Proof. The desired inequality is the statement that $A^{T,T}$ is positive semidefinite. For the second statement notice that the star tree, where the root is the direct parent of all the leaves, has $\frac{1}{2}D = \mathbf{1}_{n \times n} - I_{n \times n}$. Therefore $\lambda \mathbf{1}_{n \times n} - \frac{1}{2}D = I_{n \times n} - (1 - \lambda)\mathbf{1}_{n \times n}$, whose eigenvalues are $1, \dots, 1, 1 - (1 - \lambda)n$. The smallest λ that makes this matrix positive semidefinite is $1 - \frac{1}{n}$, as desired. \square

2.2. Equality cases. For an ultrametric tree T of radius 1 with n leaves and a nonnegative real number c , we consider the $n \times n$ matrix $A(c) = c\mathbf{1}_{n \times n} - \frac{1}{2}D$. The matrix $A(0)$ is not positive semidefinite, while the matrix $A(1 - \frac{1}{n})$ is positive semidefinite by Theorem 1.8. Since the cone of positive semidefinite matrices is closed, there is a smallest positive constant $c = c_T \leq 1 - \frac{1}{n}$ for which the matrix $A(c)$ is positive semidefinite. When do we have $c_T = 1 - \frac{1}{n}$?

We saw in the proof of Theorem 1.8 that star trees are optimal in this sense. It turns out that they are essentially the only possibility for $c_T = 1 - \frac{1}{n}$. In order to formulate the precise statement, we introduce some auxiliary definitions for ultrametric trees of radius 1. Since we allow edges with length zero, the distance function on T may define only a pseudometric on the leaves. In other words, we may have $d(i, j) = 0$ for distinct i and j .

- (1) We say that T is *leaf-positive* if every edge of T adjacent to a leaf has positive length. Note that this is equivalent to saying that for any vertex i of T , $H(i) = 0$ if and only if i is a leaf.
- (2) We say that T is a *star-metric* if, for any two leaves i, j of T , the height of their lowest common ancestor $H(i \vee j)$ is either 0 or 1.

The leaf-positive condition implies that the tree distance function is a metric. The star-metric condition characterizes those ultrametric trees that give star trees after contracting all length 0 edges and suppressing all non-root degree 2 vertices.

Proposition 2.3. [Equality case of Theorem 1.8] Let T be a leaf-positive ultrametric tree of radius 1. Then $c_T = 1 - \frac{1}{n}$ if and only if T is a star-metric.

The proof is obtained by carefully retracing the proof of Proposition 2.2.³ Proposition 2.3 will not be used in the rest of the paper.

3. LORENTZIAN POLYNOMIALS FROM TREES

3.1. Lorentzian polynomials and discrete convex analysis. Let H_n^d be the space of degree d homogeneous polynomials in n variables $x = (x_i)_{i \in E}$ with real coefficients, equipped with the usual topology of a finite-dimensional real vector space. We write ∂_i for the differential operator $\frac{\partial}{\partial x_i}$ for $i \in E$, and set

$$\partial^\alpha := \prod_{i \in E} \partial_i^{\alpha_i} \text{ for any } \alpha = (\alpha_i)_{i \in E} \in \mathbb{Z}_{\geq 0}^E.$$

In degree 2, the set of *strictly Lorentzian polynomials* $\mathring{L}_n^2 \subseteq H_n^2$ is, by definition, the set of quadratic forms f whose Hessian is entrywise positive and has Lorentzian signature:

$$\mathring{L}_n^2 := \left\{ f \in H_n^2 \mid (\partial_i \partial_j f)_{i,j \in E} \text{ has only positive entries and signature } (+, - \dots, -) \right\}.$$

In degree $d > 2$, we define the set of *strictly Lorentzian polynomials* $\mathring{L}_n^d \subseteq H_n^d$ recursively by setting

$$\mathring{L}_n^d := \left\{ f \in H_n^d \mid \partial_i f \in \mathring{L}_n^{d-1} \text{ for all } i \in E \right\}.$$

The set of *Lorentzian polynomials* L_n^d is defined to be the closure of \mathring{L}_n^d in H_n^d . Here we collect some properties of Lorentzian polynomials relevant to this paper:

- (1) If $f(x) \in L_n^d$ and A is an $n \times m$ nonnegative matrix, then $f(Ay) \in L_m^d$ [BH20, Theorem 2.10].
- (2) If $f \in L_n^d$, then $\sum_i a_i \partial_i f \in L_n^{d-1}$ for any $a_i \geq 0$ [BH20, Corollary 2.11].
- (3) If $f \in L_n^d$ and $g \in L_n^e$, then $fg \in L_n^{d+e}$ [BH20, Corollary 2.32].
- (4) A degree d bivariate polynomial $f(x, y) = \sum_{k=0}^d a_k x^k y^{d-k}$ is Lorentzian if and only if

$$(\partial_x^d f, \dots, \partial_x^{d-k} \partial_y^k f, \dots, \partial_y^d f) = \left(\frac{a_0}{\binom{d}{0}}, \dots, \frac{a_k}{\binom{d}{k}}, \dots, \frac{a_d}{\binom{d}{d}} \right)$$

is a log-concave sequence of nonnegative numbers with no internal zeros [BH20, Example 2.3].

³For details see <http://sergioc147.github.io/files/ACDEHW-equality.pdf>.

We briefly review the theory of Lorentzian polynomials and its relation to discrete convex analysis. For further details, we refer to [BH20] for Lorentzian polynomials and [Mur03] for discrete convex analysis. We consider the rank d simplex in $\mathbb{Z}_{\geq 0}^E$ defined by

$$\Delta_n^d := \{\alpha \in \mathbb{Z}_{\geq 0}^E \mid \alpha_1 + \cdots + \alpha_n = d\}.$$

For $i \in E$, we write e_i for the i -th standard basis vector of \mathbb{Z}^E . An M -convex set of rank d on E is a subset $J \subseteq \Delta_n^d$ satisfying the *symmetric exchange property*:

For any $\alpha, \beta \in J$ and $i \in E$ such that $\alpha_i > \beta_i$, there is $j \in E$ such that $\beta_j > \alpha_j$ for which both $\alpha + e_j - e_i$ and $\beta + e_i - e_j$ belong to J .

A function $\nu: \Delta_n^d \rightarrow \mathbb{R} \cup \{\infty\}$ is said to be M -convex if, for any $\alpha, \beta \in \Delta_n^d$ and $i \in E$ such that $\alpha_i > \beta_i$, there is $j \in E$ such that $\beta_j > \alpha_j$ and

$$\nu(\alpha) + \nu(\beta) \geq \nu(\alpha + e_j - e_i) + \nu(\beta + e_i - e_j).$$

Equivalently, a set of lattice points is M -convex if and only if it is the set of lattice points in an integral generalized permutahedron, and a function is M -convex if and only if it induces a regular subdivision of an integral generalized permutahedron into integral generalized permutahedra.

We say that ν is M -concave if $-\nu$ is M -convex. The *effective domain* of an M -convex function is

$$\text{dom}(\nu) := \{\alpha \in \Delta_n^d \mid \nu(\alpha) \neq \infty\},$$

and the effective domain of an M -concave function is defined similarly. Note that the effective domains of M -convex functions and M -concave functions are M -convex. The *support* of a degree d homogeneous polynomial in n variables $f = \sum_{\alpha \in \Delta_n^d} c_\alpha x^\alpha$ is defined by

$$\text{supp}(f) := \{\alpha \in \Delta_n^d \mid c_\alpha \neq 0\}.$$

The following characterization of Lorentzian polynomials is central to their relationship to matroids and M -convexity [BH20, Theorem 2.25].

Theorem 3.1. The following conditions are equivalent for $f \in \mathbb{H}_n^d$:

- (1) The polynomial f is Lorentzian.
- (2) The support of f is an M -convex set and, for all $\alpha \in \Delta_n^{d-2}$, the Hessian of $\partial^\alpha f$ has only nonnegative entries and has at most one positive eigenvalue.

From this, one may deduce the following characterizations of M -convex functions [BH20, Theorem 3.14]: A function $\nu: \Delta_n^d \rightarrow \mathbb{R} \cup \{\infty\}$ is M -convex if and only if its *normalized generating polynomial*

$$f_{q,\nu}(x) := \sum_{\alpha \in \text{dom}(\nu)} q^{\nu(\alpha)} \frac{x^\alpha}{\alpha!}$$

is a Lorentzian polynomial for all positive $q \leq 1$, where $x^\alpha / \alpha!$ is the product of $x_i^{\alpha_i} / \alpha_i!$ for $i \in E$. This in turn implies the following characterization of M -convex sets [BH20, Theorem 3.10]: A

subset J of Δ_n^d is M-convex if and only if its normalized generating polynomial

$$f_J := \sum_{\alpha \in J} \frac{x^\alpha}{\alpha!}$$

is a Lorentzian polynomial. For example, a collection of d -element subsets of E is the set of bases of a matroid on E if and only if its generating polynomial is Lorentzian.

Remark 3.2. The above characterization of M-convex functions specializes to the following characterization of M^{\natural} -concave functions on 2^E : A function $\nu : 2^E \rightarrow \mathbb{R} \cup \{-\infty\}$ is M^{\natural} -concave if and only if the *normalized homogeneous generating polynomial*

$$N(Z_{q,\nu}) := \sum_{S \subseteq E} q^{-\nu(S)} x^S \frac{y^{|E \setminus S|}}{|E \setminus S|!}$$

is a Lorentzian polynomial for all positive $q \leq 1$, where y is a homogenizing variable different from x_i for $i \in E$. Identifying the variables x_i with each other, this characterization gives the property (M2) for $I_{q,\nu;k}$. It is interesting to compare this characterization of M^{\natural} -concave functions on 2^E with that in Theorem 1.5, which implies the property (M3) for $I_{q,\nu;k}$. Note that, in general, if we omit any normalizing factor $\frac{1}{\alpha_i!}$ from the normalized generating polynomial of an M-convex function, we do not get a Lorentzian polynomial. For example, among the three bivariate quadratic forms

$$\frac{x_1^2}{2} + x_1x_2 + \frac{x_2^2}{2}, \quad x_1^2 + x_1x_2 + \frac{x_2^2}{2}, \quad x_1^2 + x_1x_2 + x_2^2,$$

only the first is a Lorentzian polynomial.

3.2. Lorentzian property of the homogeneous generating polynomial $Z_{q,\nu}$. The following lemma is a variation of known relations between tree metrics, ultrametrics, and valuated matroids. See, for instance, [MS15, Section 4.3].

Lemma 3.3. Let ν be an M^{\natural} -concave function on 2^E whose effective domain contains all subsets of E with at most 2 elements. Then, for any $0 < q \leq 1$, the function

$$d(i, j) := \begin{cases} 2q^{-\nu(i,j) + \nu(i) + \nu(j) - \nu(\emptyset)}, & \text{if } i \neq j, \\ 0, & \text{if } i = j, \end{cases}$$

defines an ultrametric on E of radius ≤ 1 .

Proof. By the local exchange property for M^{\natural} -concave functions [Mur03, Theorem 6.4], for any distinct elements i, j, k in E , the maximum among the three numbers

$$\nu(j, k) + \nu(i), \quad \nu(i, k) + \nu(j), \quad \nu(i, j) + \nu(k),$$

is achieved at least twice. Thus, the maximum among the three numbers

$$\nu(j, k) - \nu(j) - \nu(k) + \nu(\emptyset) \quad \nu(i, k) - \nu(i) - \nu(k) + \nu(\emptyset), \quad \nu(i, j) - \nu(i) - \nu(j) + \nu(\emptyset),$$

is achieved at least twice as well. Since $0 < q \leq 1$, we get that d is an ultrametric on E . Also, the exchange property for ν gives

$$\nu(i, j) + \nu(\emptyset) \leq \nu(i) + \nu(j) \text{ for any distinct } i, j \text{ in } E.$$

which implies that d has radius ≤ 1 . \square

We now prove Theorem 1.5 and deduce Theorem 1.4.

Theorem 1.5. A function $\nu : 2^E \rightarrow \mathbb{R} \cup \{-\infty\}$ is M^\natural -concave if and only if its homogeneous generating polynomial $Z_{q,\nu}$ is a Lorentzian polynomial for all positive $q \leq 1$.

Proof. The “if” direction is [EH20, Proposition 5.5]. We show that $Z_{q,\nu}$ is Lorentzian for positive $q \leq 1$ when ν is M^\natural -concave.

We observe that any M^\natural -concave function on 2^E can be approximated by a sequence of M^\natural -concave functions whose effective domains are 2^E . More precisely, if ν is such an M^\natural -concave function, then there is a sequence M^\natural -concave functions ν_k with $\text{dom}(\nu_k) = 2^E$ such that

$$\lim_{k \rightarrow \infty} \nu_k(S) = \nu(S) \text{ for every } S \subseteq E.$$

This is a special case of [BH20, Lemma 3.27], because restrictions of M -concave functions to coordinate half-spaces are M -concave [Mur03, Section 6.4]. Since L_n^d is closed, we may assume that $0 < q < 1$, and for any such q , we have

$$\lim_{k \rightarrow \infty} Z_{q,\nu_k} = Z_{q,\nu}.$$

Therefore, we may assume that $0 < q < 1$ and the effective domain of ν is 2^E .

Since the support of $Z_{q,\nu}$ is M -convex, by Theorem 3.1, it is enough to show that all the $n+1$ partial derivatives of $Z_{q,\nu}$ are Lorentzian. For any $i \in E$, we have

$$\frac{\partial}{\partial x_i} Z_{q,\nu} = Z_{q,\nu/i},$$

where ν/i is the M^\natural -concave function on $2^{E \setminus i}$ defined by $\nu/i(S) = \nu(S \cup i)$. For general discussion about the contraction ν/i of an M^\natural -concave function ν , see [Mur03, Section 6.4]. Thus, modulo induction on n , we only need to consider the Lorentzian property of the quadratic form

$$\left(\frac{\partial}{\partial y} \right)^{n-2} Z_{q,\nu} = (n-2)! \left[\frac{n(n-1)}{2} q^{-\nu(\emptyset)} y^2 + (n-1) \sum_{i \in \binom{E}{1}} q^{-\nu(i)} x_i y + \sum_{ij \in \binom{E}{2}} q^{-\nu(i,j)} x_i x_j \right].$$

Up to positive rescaling of rows and columns, the Hessian of this quadratic form is

$$A(\nu) := \left[\begin{array}{c|ccc} n/(n-1)q^{-\nu(\emptyset)} & q^{-\nu(1)} & \dots & q^{-\nu(n)} \\ \hline q^{-\nu(1)} & 0 & \dots & q^{-\nu(i,j)} \\ \vdots & & \ddots & \\ q^{-\nu(n)} & q^{-\nu(i,j)} & & 0 \end{array} \right].$$

Our goal is to show that $A(\nu)$ has at most one positive eigenvalue for any positive $q \leq 1$. Computing the Schur complement with respect to the block structure indicated above, we may block diagonalize $A(\nu)$ to

$$\left[\begin{array}{c|c} n/(n-1)q^{-\nu(\emptyset)} & 0 \\ \hline 0 & B(\nu) \end{array} \right],$$

where $B(\nu)$ is the symmetric matrix with rows and columns labeled by E and with entries

$$B(\nu)_{ij} = \begin{cases} \frac{1-n}{n}q^{\nu(\emptyset)-\nu(i)-\nu(j)} + q^{-\nu(ij)}, & \text{if } i \neq j, \\ \frac{1-n}{n}q^{\nu(\emptyset)-2\nu(i)}, & \text{if } i = j. \end{cases}$$

Thus, it is enough to show that $B(\nu)$ is negative semidefinite for positive $q \leq 1$. Rescaling rows and columns of $B(\nu)$, we obtain a matrix $C(\nu)$ with entries

$$C(\nu)_{ij} = \begin{cases} -(1 - \frac{1}{n}) + q^{-\nu(i,j)+\nu(i)+\nu(j)-\nu(\emptyset)}, & \text{if } i \neq j, \\ -(1 - \frac{1}{n}), & \text{if } i = j. \end{cases}$$

By Lemma 3.3, $d(i, j) = 2q^{-\nu(i,j)+\nu(i)+\nu(j)-\nu(\emptyset)}$ is an ultrametric on E . Since d has radius ≤ 1 by the same lemma, Theorem 1.8 implies that $C(\nu)$ is negative semidefinite for positive $q \leq 1$. This implies the same for $B(\nu)$, and hence $A(\nu)$ has at most one positive eigenvalue for positive $q \leq 1$. This finishes the proof that $Z_{q,\nu}$ is Lorentzian for positive $q \leq 1$ when ν is M^{\natural} -concave. \square

Theorem 1.4. For any M^{\natural} -concave function $\nu: 2^E \rightarrow \mathbb{R} \cup \{-\infty\}$, we have

$$I_{q,\nu;k}^2 \geq \left(1 + \frac{1}{k}\right) \left(1 + \frac{1}{n-k}\right) I_{q,\nu;k-1} I_{q,\nu;k+1} \quad \text{for all } 0 < k < n \text{ and } 0 < q \leq 1.$$

Proof. Identifying the variables x_i with each other in $Z_{q,\nu}(x, y)$, we get a bivariate polynomial with coefficients $I_{q,\nu;k}$. Since $Z_{q,\nu}$ is Lorentzian by Theorem 1.5, the specialization is Lorentzian as well [BH20, Theorem 2.10], and hence the sequence $I_{q,\nu;k}$ is ultra log-concave in k for positive $q \leq 1$ [BH20, Example 2.3]. \square

3.3. Polynomial log concavity. We prove Theorem 1.6 using the following observation.

Lemma 3.4. Let f be a homogeneous polynomial of degree d in n variables

$$f(x) = f_0(x_2, \dots, x_n) + x_1 f_1(x_2, \dots, x_n) + \dots + x_1^d f_d(x_2, \dots, x_n).$$

If f is Lorentzian, then f_i is Lorentzian for each $0 \leq i \leq d$.

Proof. The polynomial $\partial_1^i f$ is Lorentzian [BH20, Corollary 2.11], and hence its specialization $i! f_i$ is Lorentzian as well [BH20, Theorem 2.10]. \square

For any $\nu: 2^E \rightarrow \mathbb{R} \cup \{-\infty\}$, nonnegative integers a and b , and a multiset S on E , we set

$$N_{q,\nu}^S(a, b) := \sum_{|A|=a, |B|=b, S=A+B} q^{-\nu(A)-\nu(B)},$$

where the sum is over all a -element subset A of E and b -element subset B of E whose multiset union is S .

Theorem 1.6. For any M^{\natural} -concave function $\nu : 2^E \rightarrow \mathbb{R} \cup \{-\infty\}$, we have the coefficientwise inequality

$$I_{q,\nu;k}(\mathbf{x})^2 \succeq \left(1 + \frac{1}{k}\right) I_{q,\nu;k-1}(\mathbf{x}) I_{q,\nu;k+1}(\mathbf{x}) \text{ for all } 0 < k < n \text{ and } 0 < q \leq 1.$$

Proof. Fix a positive real parameter $q \leq 1$, and consider two copies of homogeneous generating polynomial of an M^{\natural} -concave function ν , say $Z_{q,\nu}(\mathbf{x}, y)$ and $Z_{q,\nu}(\mathbf{x}, z)$, where we use distinct homogenizing variables y and z for the same set of n variables \mathbf{x} . Since $Z_{q,\nu}(\mathbf{x}, y)$ and $Z_{q,\nu}(\mathbf{x}, z)$ are Lorentzian by Theorem 1.5, their product is Lorentzian too [BH20, Corollary 2.32]. Writing n for the cardinality of E as before, we have

$$Z_{q,\nu}(\mathbf{x}, y) Z_{q,\nu}(\mathbf{x}, z) = \sum_S \left[\sum_{k+j=|S|} N_{q,\nu}^S(j, k) y^{n-j} z^{n-k} \right] \mathbf{x}^S,$$

where the sum is over all multisets S on E and $|S|$ is the cardinality of S counting multiplicities. Iterating Lemma 3.4, we see that, for each S , the bivariate polynomial

$$\sum_{j+k=|S|} N_{q,\nu}^S(j, k) y^{n-j} z^{n-k}$$

is Lorentzian. Therefore, the sequence of normalized coefficients $(n-j)! (n-k)! N_{q,\nu}^S(j, k)$ is log-concave and has no internal zeros [BH20, Example 2.3]. It follows that, for all $i \leq j \leq k \leq l$ for with $i+l = j+k = |S|$, we have

$$(3.1) \quad (n-j)! (n-k)! N_{q,\nu}^S(j, k) \geq (n-i)! (n-l)! N_{q,\nu}^S(i, l).$$

This is not quite the inequality that we want, but we can apply it to a different M^{\natural} -concave function ν' constructed from ν and S to deduce the desired inequality

$$j! k! N_{q,\nu}^S(j, k) \geq i! l! N_{q,\nu}^S(i, l).$$

This will suffice, as the above displayed inequality for all S is equivalent to the coefficient-wise inequality

$$j! k! I_{q,\nu;j}(\mathbf{x}) I_{q,\nu;k}(\mathbf{x}) \succeq i! l! I_{q,\nu;i}(\mathbf{x}) I_{q,\nu;l}(\mathbf{x}).$$

Let S be a multiset on E where each element of E appears at most twice, and let E' be the set of size $n' := |E'| = |S|$ obtained from the underlying set \underline{S} of S by adding a second copy of each element that appears twice in S . We define an M^{\natural} -concave function $\nu' : 2^{E'} \rightarrow \mathbb{R} \cup \{-\infty\}$ by setting

$$\nu'(I) = \begin{cases} \nu(I), & \text{if } I \text{ is a subset of } \underline{S}, \\ -\infty & \text{if otherwise.} \end{cases}$$

By construction, for any nonnegative integers a and b satisfying $a+b = |S|$, we have

$$N_{q,\nu}^S(a, b) = N_{q,\nu'}^S(a, b).$$

Applying (3.1) to ν we obtain, for all $i \leq j \leq k \leq l$ for with $i + l = j + k = n' = |S|$, that

$$j! k! N_{q,\nu}^S(j, k) = (n' - j)! (n' - k)! N_{q,\nu'}^{S'}(j, k) \geq (n' - i)! (n' - l)! N_{q,\nu'}^{S'}(i, l) = i! l! N_{q,\nu}^S(i, l)$$

as desired. \square

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QUEEN MARY UNIVERSITY OF LONDON AND SAN FRANCISCO STATE UNIVERSITY

Email address: federico@sfsu.edu

PRINCETON UNIVERSITY

Email address: sergio.cris@princeton.edu

UNIVERSITY OF WESTERN ONTARIO

Email address: gdenham@uwo.ca

CARNEGIE MELLON UNIVERSITY

Email address: ceur@cmu.edu

PRINCETON UNIVERSITY AND KOREA INSTITUTE FOR ADVANCED STUDY

Email address: huh@princeton.edu

UNIVERSITY OF WISCONSIN-MADISON

Email address: wang@math.wisc.edu