Lifted generalized permutahedra and composition polynomials.

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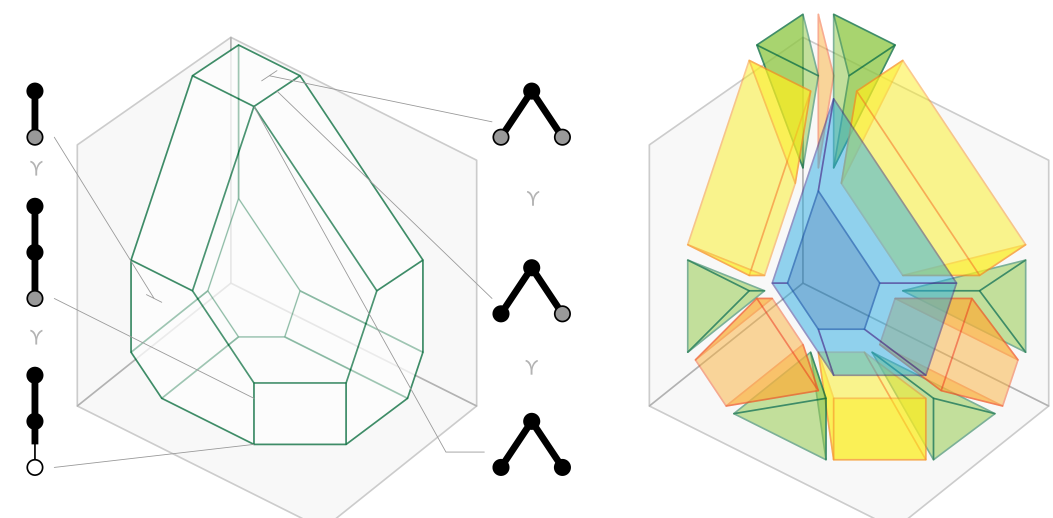
Abstract

Generalized permutahedra are the polytopes obtained from the permutahedron by changing the edge lengths while preserving the edge directions, possibly identifying vertices along the way. We introduce a lifting construction for these polytopes, which turns an n-dimensional generalized permutahedron into an $(n+1)$-dimensional one. We prove that this construction gives rise to Stasheff's multiplihedron from homotopy theory, and to the more general nestomultiplihedra, answering two questions of Devadoss and Forcey.

We construct a subdivision of any lifted generalized permutahedron whose pieces are indexed by compositions. The volume of each piece is given by a polynomial whose combinatorial properties we investigate. We show how this composition polynomial arises naturally in the polynomial interpolation of an exponential function. We prove that its coefficients are positive integers, and present evidence suggesting that they may also be unimodal.



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1 Introduction.

Generalized permutahedra are the polytopes obtained from the permutahedron by changing the edge lengths while preserving the edge directions, possibly identifying vertices along the way. These polytopes, closely related to polymatroids [12] and recently re-introduced by Postnikov [18] have been the subject of great attention due their very rich combinatorial structure. Examples include several remarkable polytopes which naturally appear in homotopy theory, in geometric group theory, and in various moduli spaces: permutahedra, matroid polytopes [4], Pitman-Stanley polytopes [17], Stasheff's associahedra [27], Carr and Devadoss's graph associahedra [5], Stasheff's multiplihedra [27], Devadoss and Forcey's multiplihedra [7], and Feichtner and Sturmfels's and Postnikov's nestohedra [9, 18].

In part 1 of the paper, we introduce a lifting construction which takes a generalized permutahedron P in $\mathbb{R}^n$ into a generalized permutahedron $P(q)$ in $\mathbb{R}^{n+1}$, where $0 \leq q \leq 1$. We show that the lifting construction connects many important generalized permutahedra:

generalized permutahedron P	lifting $P(q)$
permutahedron P_n	permutahedron P_{n+1}
associahedron $\mathcal{K}_n$	multiplihedron $\mathcal{J}_n$
graph associahedron $\mathcal{K}G$	graph multiplihedron $\mathcal{J}G$
nestohedron $\mathcal{K}\mathcal{B}$	nestomultiplihedron $\mathcal{J}\mathcal{B}$
matroid polytope P_M	independent set polytope I_M $(q = 0)$

We provide geometric realizations of these polytopes and concrete descriptions of their face lattices. In particular, we answer two questions of Devadoss and Forcey: we find the Minkowski decomposition of the graph multiplihedra $\mathcal{J}G$ into simplices, and we construct the nestomultiplihedron $\mathcal{J}\mathcal{B}$.

We also construct a subdivision of any lifted generalized permutahedron $P(q)$ whose pieces are indexed by compositions c. The volume of each piece is essentially given by a polynomial in q, which we call the *composition polynomial* $g_c(q)$.

Part 2 is devoted to the combinatorial properties of the composition polynomial $g_c(q)$ of a composition $c = (c_1, \ldots, c_k)$. We prove that $g_c(q)$ arises naturally in the polynomial interpolation of an exponential function. We also give a combinatorial interpretation of $g_c(q)$ in terms of the enumeration of linear extensions of a poset P_c. We prove that $g_c(q) = (1-q)^k f_c(q)$ where $f_c(q)$ is a polynomial with $f_c(1) \neq 0$. We prove that the coefficients of $f_c(q)$ are positive integers. We believe they may be unimodal as well; we have verified this for all 335,922 compositions of at most 7 parts and sizes of parts at most 6.

PART 1. LIFTED GENERALIZED PERMUTOHEDRA.

The first part of the paper is devoted to the lifting construction, which turns an n-dimensonal generalized permutahedron P into an $(n+1)$-dimensional one $P(q)$ which depends on a parameter $0 \leq q \leq 1$.

In Section 2 we introduce the q-lifting $P(q)$. We describe its defining inequalities, and its decomposition as a Minkowski sum of simplices. We show that all q-liftings $P(q)$ with $0 < q < 1$ are combinatorially isomorphic.

In Section 3 we study the face structure of the lifting of P. As a warmup, we show that the lifting of the permutahedron P_n is the permutahedron P_{n+1}. We then describe the face lattice of $P(q)$ in terms of the face lattice of P.

In Section 4 we begin by recalling Postnikov's and Feichtner-Sturmfels's construction of the nestohedron $\mathcal{KB}$, and their description of its face lattice in terms of $\mathcal{B}$-forests. We then show that the lifting of $\mathcal{KB}$ is the nestomultiplihedron $\mathcal{JB}$, whose face lattice we describe in terms of painted $\mathcal{B}$-forests. As special cases, we see how the multiplihedra $\mathcal{J}_n$ and the graph multiplihedra $\mathcal{J}G$ arise from the lifting construction.

In Section 5 we give a decomposition of the lifted generalized permutahedron $P(q) \subset \mathbb{R}^n$ whose pieces $P^\pi(q)$ are in bijection with the ordered partitions π of $[n]$. We show that the volume of $P^\pi(q)$ is essentially given by a polynomial in q, which is the subject of study of Part 2 of the paper.

2 Lifting a generalized permutahedron.

The *permutahedron* P_n is the polytope in $\mathbb{R}^n$ whose $n!$ vertices are the permutations of the vector $(1, 2, \ldots, n)$. A *generalized permutahedron* is a deformation of the permutahedron, obtained by moving the vertices of P_n in such a way that all edge directions and orientations are preserved, while possibly identifying vertices along the way. [20].

Postnikov showed [18] that every generalized permutahedron can be written in the form:

$$P_n(\{z_I\}) = \left\{ (t_1, \ldots, t_n) \in \mathbb{R}^n : \sum_{i=1}^{n} t_i = z_{[n]}, \sum_{i \in I} t_i \geq z_I \text{ for all } I \subseteq [n] \right\}$$

where z_I is a real number for each $I \subseteq [n] := \{1, \ldots, n\}$, and $z_\emptyset = 0$. The following characterization was announced by Morton et. al. [16, Theorem 17] and Postnikov [19]. A complete proof is written down in [1]; see also [21].

Theorem 2.1. *A set of parameters $\{z_I\}$ defines a generalized permutahedron $P_n(\{z_I\})$ if and only if the z_I satisfy the supermodular inequalities for all $I, J \subseteq [n]$:*

$$z_I + z_J \leq z_{I \cup J} + z_{I \cap J}.$$

Remark 2.2. By performing a parallel shift, we will assume that all our generalized permutahedra are in the positive orthant. In particular, this implies that $z_I \geq 0$ for all $I \subseteq [n]$, and that $z_I \leq z_J$ for $I \subseteq J \subseteq [n]$.

We now introduce *lifting*, a procedure which converts a generalized permutahedron in $\mathbb{R}^n$ into a *lifted generalized permutahedron* in $\mathbb{R}^{n+1}$.

Definition 2.3. *Given a generalized permutahedron $P = P_n(\{z_I\})$ in $\mathbb{R}^n$ and a number $0 \le q \le 1$, define the q-*lifting *of P to be the polytope $P(q)$ given by the inequalities*

$$\sum_{i=1}^{n+1} t_i = z_{[n]}, \qquad \sum_{i\in I} t_i \ge q z_I \text{ for } I \subseteq [n], \qquad \sum_{i\in I\cup\{n+1\}} t_i \ge z_I \text{ for } I \subseteq [n].$$

In other words, $P(q) := P_{n+1}(\{z'_I\})$ where $z'_J = qz_J$ and $z'_{J\cup\{n+1\}} = z_J$ for $J \subseteq [n]$. The polytope $P(q)$ is called a lifted generalized permutahedron*.*

We will let the lifting *of P refer to any q-lifting with $0 < q < 1$. We will see in Corollary 2.6 that all such q-liftings are combinatorially isomorphic.*

Proposition 2.4. *If P is a generalized permutahedron, then its q-lifting $P(q)$ is a generalized permutahedron.*

Proof. One easily checks that the hyperplane parameters $\{z'_I\}_{I\subseteq[n+1]}$ are supermodular. $\square$

Notice that the 1-lifting $P(1)$ is the natural embedding of P in the hyperplane $x_{n+1} = 0$ of $\mathbb{R}^{n+1}$. The 0-lifting $P(0) = P_{n+1}(\{z'_I\})$ is the generalized permutahedron in $\mathbb{R}^{n+1}$ defined by $z'_J = 0$ and $z'_{J\cup\{n+1\}} = z'_J$ for all $J \subseteq [n]$.

Recall that the *Minkowski sum* of two polytopes P and Q in $\mathbb{R}^n$ is defined to be $P + Q := \{p + q \,:\, p \in P, q \in Q\}$. Since the hyperplane parameters $\{z_I\}$ of generalized permutahedra are additive with respect to Minkowski sums [2, 18], we have:

Proposition 2.5. *For $0 \le q \le 1$, the q-lifting of any generalized permutahedron P satisfies $P(q) = qP(1) + (1 - q)P(0)$.*

Corollary 2.6. *All q-liftings of P with $0 < q < 1$ are combinatorially isomorphic.*

Proof. By Proposition 2.5, the normal fan of $P(q)$ is the common refinement of the normal fans of $P(0)$ and $P(1)$. $\square$

For each $I \subseteq [n]$, consider the simplex $\Delta_I = \mathrm{conv}\{e_i \,:\, i \in I\}$. Any generalized permutahedron $P = P_n(\{z_I\})$ can be written uniquely as a signed Minkowski sum of simplices in the form $P = P_n(\{y_I\}) := \sum y_I \Delta_I$ for $y_I \in \mathbb{R}$.[1] [2,18] The z-parameters and the y-parameters of P are linearly related by the equations

$$z_I = \sum_{J\subseteq I} y_J, \qquad \text{for all } I \subseteq [n].$$

Proposition 2.7. *The q-lifting of the generalized permutahedon $P = \sum_I y_I \Delta_I$ is*

$$P(q) = q\sum_I y_I \Delta_I + (1 - q)\sum_I y_I \Delta_{I\cup\{n+1\}}.$$

[1]An equation like $P - Q = R$ should be interpreted as $P = Q + R$.

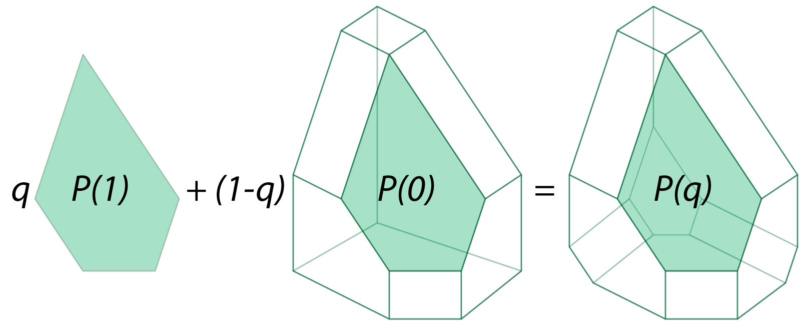

Figure 1: The q-lifting of a generalized permutahedron $P_n(\{y_I\})$, shown projected onto the 3-dimensional hyperplane $x_4 = 0$.

Proof. This follows directly from the linear relation between the z_I and the y_I. □

From these observations it follows that the face of $P(q)$ maximized in the direction $(1, \ldots, 1, 0)$ is a copy of P, while the face maximized in the opposite direction is a copy of P scaled by q. The vertices of $P(q)$ will come from vertices of P, with a factor of q applied to certain specific coordinates. We describe them in Section 5.

3 Faces of lifted generalized permutahedra.

We now look into the face structure of lifted generalized permutahedra. An important initial observation is that their face lattices are always coarsenings of the face lattice of the permutahedron P_n. [16, 18, 20]

Definition 3.1. *Consider the linear functional $f(x_1, \ldots, x_n) = a_1x_1 + \cdots + a_nx_n$. We partition $[n]$ into blocks $A_1, \ldots, A_k$ such that $a_i = a_j$ if and only if i and j both belong to the same block A_s, and $a_i < a_j$ if and only if $i \in A_s$ and $j \in A_t$ for some $s < t$. If we let $\pi = A_1|\cdots|A_k$ then we say that the functional f is of* type π. *Slightly abusing notation, we write $f(x)$ as $f_\pi(x)$. For a generalized permutahedron P in $\mathbb{R}^n$, the face of P maximizing f only depends on π, and we call it P_π.*

Recall that the face lattice $\mathcal{L}(P_n)$ of the permutahedron P_n is isomorphic to the poset $(\mathcal{P}^n, \prec)$, where $\mathcal{P}^n$ is the set of all ordered partitions of the set $[n]$, and $\pi \prec \pi'$ if and only if π' coarsens π [18]. First we show that the q-lifted permutahedron $P_n(q)$ is combinatorially equivalent to P_{n+1}.

Proposition 3.2. *The lifting of the permutahedron P_n is combinatorially equivalent to the permutahedron P_{n+1}.*

Proof. By definition $P_n(q)$ is a generalized permutahedron in $\mathbb{R}^{n+1}$, and hence its face lattice is a coarsening of the poset of ordered partitions on a set of size $n + 1$. We will show that this coarsening is trivial; *i.e.*, that every strict containment of faces in P_{n+1} corresponds to a strict containment of faces in $P_n(q)$.

The permutahedron P_n is a zonotope, and it can be represented as the Minkowski sum of all coordinate 1-simplices Δ_{ij} for $1 \leq i < j \leq n$. Using our established notation, we write $P_n = P_n(\{y_I\})$ where $y_I = 1$ if I has size 2, and $y_I = 0$ otherwise. Let $\pi = B_1|\cdots|B_k$ be an ordered partition of $[n+1]$, and let $P_n(q)_\pi$ be the corresponding maximal face of $P_n(q)$. It suffices to show that any minimal coarsening σ of π, obtained by joining blocks B_i and B_{i+1}, leads to a different maximal face $P_n(q)_\sigma$.

For every pair $b_1, b_2 \in [n+1]$ the Minkowski decomposition of $P_n(q)$ contains a simplex with $\Delta_{b_1 b_2}$ as a face. Take $b_1 \in B_i$ and $b_2 \in B_{i+1}$. Then the Minkowski decomposition of the face $P_n(q)_\sigma$ includes a one-dimensional contribution from $\Delta_{b_1 b_2}$, whereas the decomposition of $P_n(q)_\pi$ does not. Thus $P_n(q)_\pi$ is properly contained in $P_n(q)_\sigma$, as we wished to show. □

Now we extend our focus to face lattices of general q-liftings. For convenience let us assume that the generalized permutahedra P we are analyzing have nonempty intersection with the interior of the positive orthant of $\mathbb{R}^n$. If this is not the case, we can simply project out the unused coordinate(s) so that this condition is satisfied.

Definition 3.3. *Let P be a generalized permutahedron in $\mathbb{R}^n$, and let π and μ be ordered partitions of $[n]$. Then we say that $\pi \sim \mu$ if $P_\pi = P_\mu$. We can write the face lattice of P as*

$$\mathcal{L}(P) \cong (\mathcal{P}^n, \prec) \,/ \sim .$$

The order $\prec$ on equivalence classes is as follows: the equivalence class $[\mu]$ covers $[\pi]$ if and only if there exist $\pi' \in [\pi]$ and $\mu' \in [\mu]$ such that μ' coarsens π'.

We now describe the equivalence relation $\sim'$ on $\mathcal{P}^{n+1}$ induced by $P(q)$ in terms of the equivalence relation $\sim$ on $\mathcal{P}^n$ induced by P.

Definition 3.4. *Let $\pi = A_1|\cdots|A_{k_1}$ and $\mu = B_1|\cdots|B_{k_2}$ be ordered partitions of $[n]$. Let π' and μ' be obtained from π' and μ' by adding the element $\{n+1\}$ to the (possibly new, in which case we relabel the blocks) blocks A_{j_1} and B_{j_2}, respectively. Then we say that $\pi' \sim' \mu'$ if the following conditions hold:*

1. *$\pi \sim \mu$, and*
2. *$A_{j_1} = B_{j_2}$ and $\bigcup_{i>j_1} A_i = \bigcup_{i>j_2} B_i$.*

Proposition 3.5. *Using the notation established above, the face lattice of $P(q)$ is given by*

$$\mathcal{L}(P(q)) \cong \left(\mathcal{P}^{n+1}, \prec\right) / \sim' .$$

Proof. Write $P = P_n(\{y_I\})$. The assumption that P intersects the positive orthant implies that for every $i \in [n]$ there is some $I \subseteq [n]$ that contains i such that $y_I \neq 0$. By Proposition 2.7 we can decompose $P(q)_{\pi'}$ into

$$P(q)_{\pi'} = q \sum_{I \subseteq [n]} y_I (\Delta_I)_\pi + (1-q) \sum_{I \subseteq [n]} y_I (\Delta_{I \cup \{n+1\}})_{\pi'}.$$

where we write $(\Delta_I)_\pi$ instead of $(\Delta_I)_{\pi'}$ because all of the Δ_I are on the hyperplane $x_{n+1} = 0$. The decomposition for $P(q)_{\sigma'}$ is analogous. Now if condition 1 of Definition 3.4 is not satisfied, then in the above expression the sums $\sum_{I\subseteq[n]} y_I(\Delta_I)_\pi$ and $\sum_{I\subseteq[n]} y_I(\Delta_I)_\sigma$ will be unequal. If condition 2 is not satisfied, then similarly $(\Delta_{I\cup\{n+1\}})_{\pi'}$ and $(\Delta_{I\cup\{n+1\}})_{\sigma'}$ must differ for some I with $y_I \neq 0$ by our positive orthant assumption. The reader can verify that both of these implications are reversible. □

4 Nestohedra and nestomultiplihedra.

In his work on homotopy associativity for A_∞ spaces, Stasheff [27] defined the *multiplihedron* $\mathcal{J}_n$, a cell complex which has since been realized in different geometric contexts by Fukaya, Oh, Ohta, and Ono [11], by Mau and Woodward [15], and others. It was first realized as a polytope by Forcey [8].

More generally, Devadoss and Forcey [7] defined, for each graph G, the *graph multiplihedron* $\mathcal{J}G$. This is a polytope related to the graph associahedron $\mathcal{K}G$. [3,5] When G has no edges, they gave a description of $\mathcal{J}G$ as a Minkowski sum. They asked for a Minkowski sum description of $\mathcal{J}G$ for arbitrary G.

In a different direction, Postnikov [18] defined the *nestohedron* $\mathcal{K}B$, an extension of graph associahedra to the more general context of building sets $\mathcal{B}$. Devadoss and Forcey asked whether there is a notion of *nestomultiplihedron* $\mathcal{J}\mathcal{B}$, which extends the graph multiplihedra to this context.

In this section we answer these questions affirmatively in a unified way, by showing that the q-lifting of the graph associahedron $\mathcal{K}G$ is the graph multiplihedron $\mathcal{J}G$ and, more generally, the q-lifting of the nestohedron $\mathcal{K}B$ is the desired nestomultiplihedron $\mathcal{J}\mathcal{B}$.

4.1 Nestohedra and $\mathcal{B}$-forests.

Definition 4.2. *[9, 18] A* building set *$\mathcal{B}$ on a ground set $[n]$ is a collection of subsets of $[n]$ such that:*
(B1) If $I, J \in \mathcal{B}$ and $I \cap J \neq \emptyset$ then $I \cup J \in \mathcal{B}$.
(B2) For every $e \in [n]$, $\{e\} \in \mathcal{B}$.

An important example is the following: given a graph G on a vertex set $[n]$, the associated building set $\mathcal{B}(G)$ consists of the subsets $I \subseteq [n]$ for which the induced subgraph $G|_I$ is connected. Such subsets are sometimes called the *tubes* of G.

If $\mathcal{B}$ is a building set on $[n]$ and $A \subseteq [n]$, define the *induced building set* of $\mathcal{B}$ on A to be $\mathcal{B}|_A := \{I \in \mathcal{B} : I \subseteq A\}$. Also let $\mathcal{B}_{\max}$ be the set of containment-maximal elements of $\mathcal{B}$.

Definition 4.3. *[9, 18] A* nested set *$\mathcal{N}$ for a building set $\mathcal{B}$ is a subset $\mathcal{N} \subseteq \mathcal{B}$ such that:*
(N1) If $I, J \in \mathcal{N}$ then $I \subseteq J$ or $J \subseteq I$ or $I \cap J = \emptyset$.
(N2) If $J_1, \ldots, J_k \in \mathcal{N}$ are pairwise incomparable and $k \geq 2$ then $J_1 \cup \cdots \cup J_k \notin \mathcal{B}$.
(N3) $\mathcal{B}_{\max} \subseteq \mathcal{N}$.

The nested set complex $\mathcal{N}(\mathcal{B})$ *of* $\mathcal{B}$ *is the simplicial complex on* $\mathcal{B}$ *whose faces are the nested sets of* $\mathcal{B}$*.*

When $\mathcal{B}(G)$ is the building set of tubes of a graph, the nested sets are called the *tubings* of G. If G is the graph shown in Figure 2(a), an example of a nested set or tubing is $\mathcal{N} = \{3, 4, 6, 7, 379, 48, 135679, 123456789\}$, shown in Figure 2(b).[2]

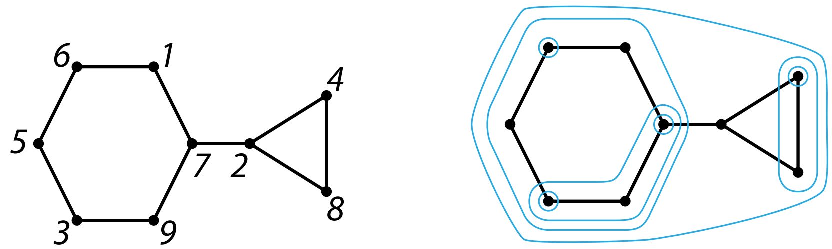

Figure 2: (a) A graph G. (b) A nested set or tubing of G.

The sets in a nested set $\mathcal{N}$ form a poset by containment. This poset is a forest rooted at $\mathcal{B}_{\max}$ by (N1). Relabelling each node N with the set $\widehat{N} := N \setminus \bigcup_{M \in \mathcal{N}\,:\, M<N} M$, we obtain a $\mathcal{B}$-forest:

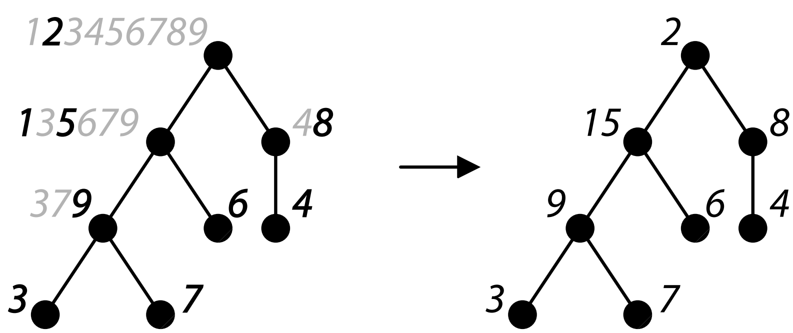

Figure 3: The poset and the $\mathcal{B}$-forest for the nested set $\mathcal{N} = \{3, 4, 6, 7, 379, 48, 135679, 123456789\}$ of Figure 2(b).

Definition 4.4. *[9, 18] Given a building set* $\mathcal{B}$ *on* $[n]$*, a* $\mathcal{B}$-forest $\mathcal{N}$ *is a rooted forest whose vertices are labeled with non-empty sets partitioning* $[n]$ *such that:*
(F1) For any node S*,* $\mathcal{N}_{\leq S} \in \mathcal{B}$*.*
(F2) If $S_1, \ldots, S_k$ *are incomparable and* $k \geq 2$*,* $\bigcup_{i=1}^{k} \mathcal{N}_{\leq S_i} \notin \mathcal{B}$*.*
(F3) If $R_1, \ldots, R_r$ *are the roots of* F*, then the sets* $\mathcal{N}_{\leq R_1}, \ldots, \mathcal{N}_{\leq R_r}$ *are precisely the maximal elements of* $\mathcal{B}$*.*

Here $\mathcal{N}_{\leq S} := \bigcup_{T \leq S} T$. It is clear from the definitions that nested sets for $\mathcal{B}$ are in bijection with $\mathcal{B}$-forests. As the notation suggests, we will make no distinction between a nested set and its corresponding $\mathcal{B}$-forest.

[2]We omit the brackets from the sets in $\mathcal{N}$ for clarity.

Given a $\mathcal{B}$-forest $\mathcal{N}$, the *contraction* of an edge ST (where T is directly above S in the forest) is obtained by removing the edge ST, and relabeling the resulting merged vertex with the set $S \cup T$. Containment of nested sets corresponds to successive contraction of $\mathcal{B}$-forests. Say $\mathcal{N} \leq \mathcal{N}'$ if the nested set $\mathcal{N}$ is contained in the nested set $\mathcal{N}'$ or, equivalently, if the $\mathcal{B}$-forest $\mathcal{N}$ is obtained from the $\mathcal{B}$-forest $\mathcal{N}'$ by successive contraction. Then we have:

Theorem 4.5. *[9, 18] The face poset of the* nestohedron

$$\mathcal{K}B := \sum_{B \in \mathcal{B}} \Delta_B$$

is isomorphic to the opposite of the poset of $\mathcal{B}$-forests.

It is worth remarking that the *graph associahedron* $\mathcal{K}G$ is the nestohedron for the building set $\mathcal{B}(G)$ of the graph G. For instance, if $G = P_n$ is the path with n vertices, then $\mathcal{B}(P_n) = \{[i,j] : 1 \leq i \leq j \leq n\}$ is the nested set of intervals, and the resulting nestohedron is the *associahedron* $\mathcal{K}_n$. Figure 4 illustrates the case $n = 3$. There is a simple bijection between $\mathcal{B}(P_3)$-forests and planar trees on $n+1$ unlabeled leaves. A "painted" version of this bijection is illustrated in Figure 6.

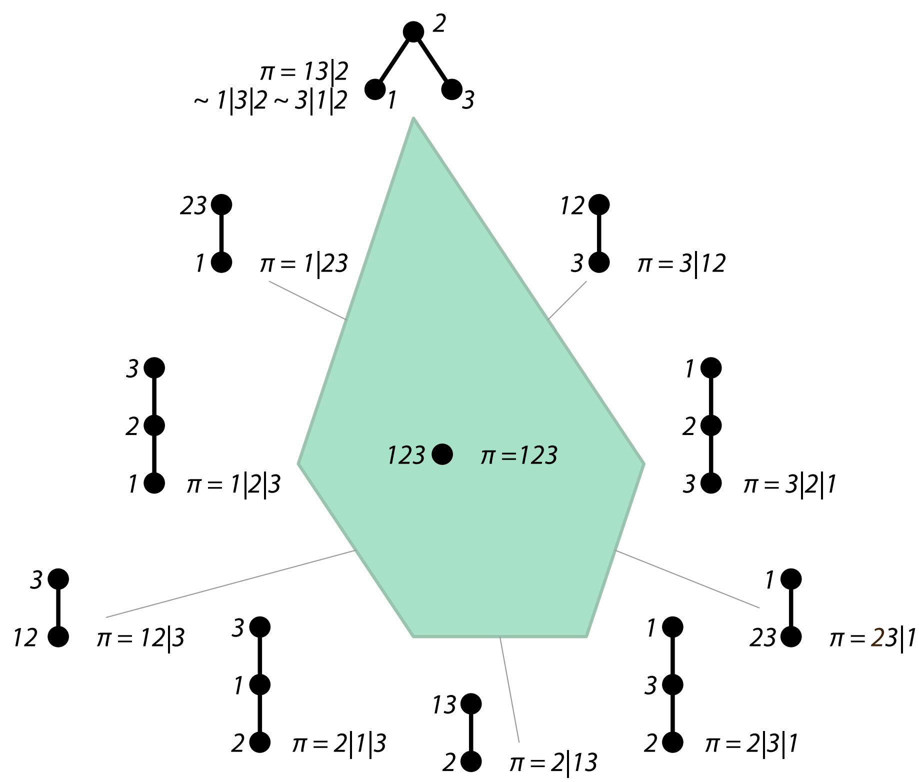

Figure 4: The associahedron $\mathcal{K}_3$, whose faces are labeled by $\mathcal{B}(P_3)$-forests. Next to each face, we have also indicated the partitions of [3] which maximize it.

4.6 Nestomultiplihedra and painted $\mathcal{B}$-forests.

Definition 4.7. *A* painted $\mathcal{B}$-forest $\overline{\mathcal{N}} = (\mathcal{N}^-, \mathcal{N}^0, \mathcal{N}^+)$ *is a $\mathcal{B}$-forest $\mathcal{N}$ together with a partition of the vertices into a downset $\mathcal{N}^-$, an antichain $\mathcal{N}^0$, and an upset $\mathcal{N}^+$ such that $\mathcal{N}^- \cup \mathcal{N}^0$ is a downset (and hence $\mathcal{N}^0 \cup \mathcal{N}^+$ is an upset). The vertices of $\mathcal{N}^-, \mathcal{N}^0$, and $\mathcal{N}^+$ are colored white, grey, and black, respectively.*

As a visual aid, we shade all half-edges above and below the black vertices, and above the grey vertices. The result is a connected "coat of paint" starting at the root of each tree in the forest. Figure 5 shows a painted $\mathcal{B}$-forest for the building set of the graph in Figure 2(a). Here $\mathcal{N}^- = \{3, 4, 6, 7\}$, $\mathcal{N}^0 = \{8, 9\}$, and $\mathcal{N}^+ = \{15, 2\}$.

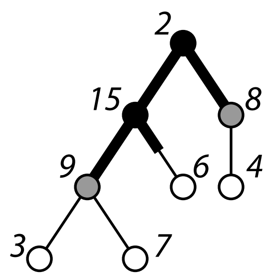

Figure 5: A painted $\mathcal{B}$-forest. The vertices in $\mathcal{N}^-, \mathcal{N}^0$, and $\mathcal{N}^+$ are shaded white, black, and grey, respectively.

This notion is compatible with the notion of painted trees in [8]. When $\mathcal{B}(P_n) = \{[i, j] : 1 \leq i \leq j \leq n\}$ is the nested set of intervals – which is also the nested set of the path P_n – the painted $\mathcal{B}(P_n)$-forests are in bijection with the *painted trees* of [8]. The bijection, which is illustrated in Figure 6, is as follows. Recall that a painted tree is planar and unlabeled. There are $n - 1$ nooks between the pairs of adjacent siblings. Travel clockwise around the tree, starting at the root, and number the nooks $1, \ldots, n-1$ in the order that they are visited. Label each internal vertex with the set of numbers of its nooks. Also color each vertex white, grey, or black, according to whether its surroundings are completely uncolored, completely colored, or half colored. Finally remove the root and all the leaves, and turn the tree upside down. The result is a painted $\mathcal{B}(P_n)$-tree, and one easily checks that this procedure is reversible.

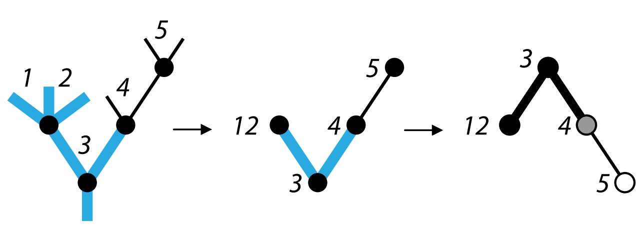

Figure 6: A painted tree and the corresponding $\mathcal{B}(P_n)$-forest.

Similarly, if $\mathcal{B}(G)$ is the building set of a graph G, then there is a natural bijection between the painted $\mathcal{B}(G)$-forests and the *marked tubings* of [7].

Given a painted $\mathcal{B}$-forest $\mathcal{N}$, the *contraction* of an edge ST is obtained by removing the edge ST and relabeling the resulting merged vertex with the set $S \cup T$. If the vertices S and T had the same color, then the new vertex $S \cup T$ is given the same color. If they had different colors, then $S \cup T$ is colored grey.

When we contract an edge whose vertices are either both black (BB), both white (WW), or grey and white (GW), we obtain a painted $\mathcal{B}$-forest. However, when we contract a BG or BW edge ST, where S is black and T is grey or white, the result may not be a painted $\mathcal{B}$-forest. To obtain one, we need to contract all BG edges ST' where T' is a grey descendent of S. We call this set of BG edges a *BG* bunch.

Definition 4.8. *Define a partial order on painted $\mathcal{B}$-forests by saying that $\mathcal{N} \leq \mathcal{N}'$ if the $\mathcal{B}$-forest $\mathcal{N}$ is obtained from the $\mathcal{B}$-forest $\mathcal{N}'$ by successively:*

- *contracting a BB, WW, or GW edge,*
- *contracting a BG bunch,*
- *converting a white or black vertex into a grey vertex.*

"Contracting a BW edge (and the corresponding BG bunch if it exists)" also brings us down in the poset of painted $\mathcal{B}$-forests, but such a contraction is a combination of the operations on the list. Therefore we do not include it.

Figure 7 shows the multiplihedron $\mathcal{J}_3$ (which is also the graph multiplihedron $\mathcal{J}K_3$, as well as the nestomultiplihedron $\mathcal{J}\mathcal{B}(K_3)$ for the building set of K_3), whose faces are in order-preserving bijective correspondence with the painted trees on [3]. Our next theorem constructs the *nestomultiplihedron*, which plays the analogous role for an arbitrary building set $\mathcal{B}$.

Theorem 4.9. *The face poset of the* nestomultiplihedron

$$\mathcal{J}\mathcal{B} := \sum_{B \in \mathcal{B}} \Delta_B + \sum_{B \in \mathcal{B}} \Delta_{B \cup \{n+1\}}$$

is isomorphic to the opposite of the poset of painted $\mathcal{B}$-forests.

Proof. Let π' be an ordered partition of $[n+1]$ and π the partition of $[n]$ obtained by removing $n+1$ from π'. Consider the face $(\mathcal{J}\mathcal{B})_{\pi'}$ maximized in direction π':

$$(\mathcal{J}\mathcal{B})_{\pi'} = \sum_{B \in \mathcal{B}} (\Delta_B)_\pi + \sum_{B \in \mathcal{B}} (\Delta_{B \cup \{n+1\}})_{\pi'}. \tag{1}$$

For each $B \in \mathcal{B}$ let $j(B)$ be the largest j for which B intersects π_j. Notice that j is a weakly increasing function, in the sense that $N \subset M$ implies $j(N) \leq j(M)$. Write $B_\pi := B \cap \pi_{j(B)}$, so $(\Delta_B)_\pi = \Delta_{B_\pi}$. Let

$$\mathcal{N} = \mathcal{N}_\pi(\mathcal{B}) := \{N \in \mathcal{B} \,:\, j(N) < j(M) \text{ for all } M \in \mathcal{B} \text{ with } N \subsetneq M\}.$$

Alternatively, construct $\mathcal{N}$ recursively by the following branching procedure: The maximal elements $N_1, \ldots, N_k$ of $\mathcal{B}_{\max}$ are in $\mathcal{N}$, and every other $B \in \mathcal{B}$ is a subset of one such $N_i \in \mathcal{B}_{\max}$. If $B \cap (N_i)_\pi \neq \emptyset$ then B is not in $\mathcal{N}$. The remaining $B \subset N_i \backslash (N_i)_\pi$ are the elements of the induced building set $\mathcal{B}|_{N_i \backslash (N_i)_\pi}$. Construct

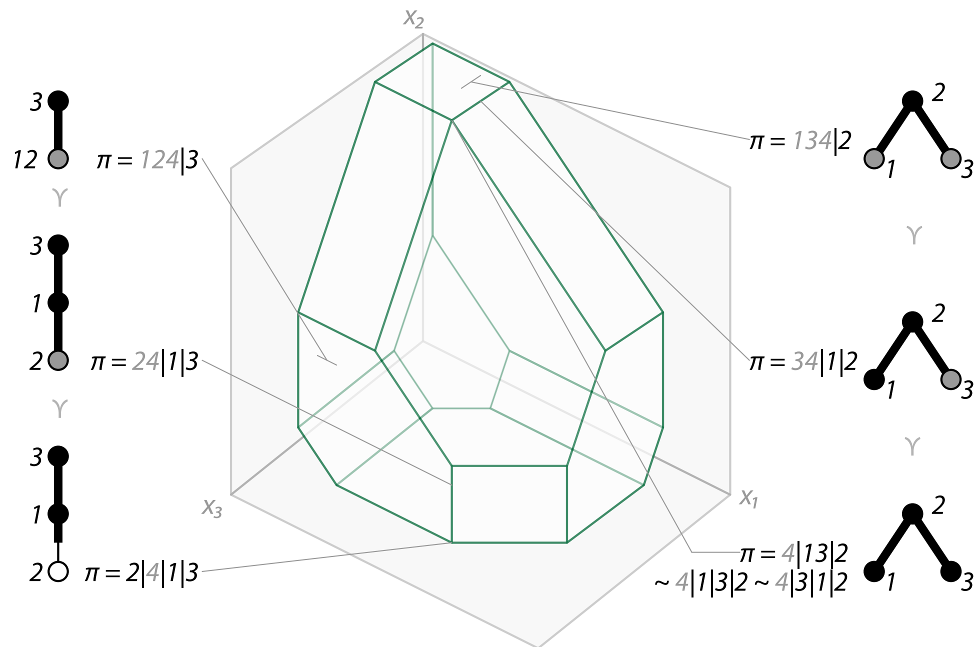

Figure 7: The multiplihedron $\mathcal{J}_3 = \mathcal{J}K_3$ (projected onto the hyperplane $x_4 = 0$), whose faces are labeled by painted $\mathcal{B}(K_3)$-forests. Next to some of the faces, we indicate the corresponding $\mathcal{B}(K_3)$-forest, as well as the partitions of [4] which maximize them.

the corresponding nested set $\mathcal{N}_i$ in each $\mathcal{B}|_{N_i \setminus (N_i)_\pi}$, and let $\mathcal{N} = \mathcal{B}_{\max} \cup \mathcal{N}_1 \cup \cdots \cup \mathcal{N}_k$. The result is a nested set.

For the building set $\mathcal{B}$ of the graph in Figure 2(a), and the ordered partition $\pi = 347|6|89|15|2$, we obtain the nested set $\mathcal{N} = \{3, 4, 6, 7, 379, 48, 135679, 123456789\}$ of Figure 2(b). Note that Figure 3 encodes the branching procedure described in the previous paragraph.

If $n+1$ was added between blocks π_i and π_{i+1} of π to be in its own block in π', let $k = i + \frac{1}{2}$. Otherwise, if $n+1$ was added to block π_i, let $k = i$. Let

$$\begin{aligned}
\mathcal{N}^+ &= \{N \in \mathcal{N} : j(N) > k\}, \\
\mathcal{N}^0 &= \{N \in \mathcal{N} : j(N) = k\}, \\
\mathcal{N}^- &= \{N \in \mathcal{N} : j(N) < k\}
\end{aligned}$$

By the definition of $\mathcal{N}$, the set $\mathcal{N}^0$ is an antichain. Also, since $j(\cdot)$ is weakly increasing, $\mathcal{N}^-$ and $\mathcal{N}^- \cup \mathcal{N}^0$ are order ideals and $\mathcal{N}^+$ is an order filter. Therefore $\overline{\mathcal{N}}_{\pi'}(\mathcal{B}) := \overline{\mathcal{N}} = (\mathcal{N}^+, \mathcal{N}^0, \mathcal{N}^-)$ is a painted $\mathcal{B}$-forest.

In the previous example, if $\pi' = 347|6|8910|15|2$ we obtain the painted $\mathcal{B}$-forest $\overline{\mathcal{N}}$ of Figure 5.

We plan to label the face $(\mathcal{J}\mathcal{B})_{\pi'}$ with the painted $\mathcal{B}$-forest $\overline{\mathcal{N}}$. In order to do that, we need to show that $\overline{\mathcal{N}}$ actually determines $(\mathcal{J}\mathcal{B})_{\pi'}$. By (1) it suffices to show that $\overline{\mathcal{N}}$ determines B_π and $(B \cup \{n+1\})_{\pi'}$ for all $B \in \mathcal{B}$. One easily checks (see [18]) that if $N \in \mathcal{N}$ then $N_\pi = N - \bigcup_{M \in \mathcal{N}\,:\, M \subsetneq N} M$, which depends only on $\mathcal{N}$. Now for an arbitrary $B \in \mathcal{B}$ let N be the minimal set in $\mathcal{N}$ containing B. From the expression for N_π above we see that $N_\pi \cap B$ is non-empty, and therefore $B_\pi = N_\pi \cap B = N \cap \pi_{j(N)} \cap B$, which only depends on $\mathcal{N}$. Finally observe that $(B \cup \{n+1\})_{\pi'}$ equals B_π if $N \in \mathcal{N}^-$, or $B_\pi \cup \{n+1\}$ if $N \in \mathcal{N}^0$, or $\{n+1\}$ if $N \in \mathcal{N}^+$, and so it is determined by $\overline{\mathcal{N}}$.

Having shown that every face is labeled by a painted $\mathcal{B}$-forest, we need to show that every painted forest $\overline{\mathcal{N}} = (\mathcal{N}^+, \mathcal{N}^0, \mathcal{N}^-)$ labels a face. Label the nodes of $\overline{\mathcal{N}}$ using all the numbers $1, 2, \ldots, m$, possibly with repetitions, strictly increasingly up the forest, in such a way that the nodes in $\mathcal{N}^-$ get labels $1, \ldots, k-1$, the nodes in $\mathcal{N}^0$ all get the label k (if $\mathcal{N}^0 \neq \emptyset$), and the nodes in $\mathcal{N}^+$ get the labels $k+1, \ldots, m$. Give $n+1$ the label k. In general there are many such labellings. Now consider the partition π' of $[n+1]$ which places the nodes labeled i in part π'_i. We claim that the face $(P_{\mathcal{B}})_{\pi'}$ is labeled by the painted $\mathcal{B}$-forest $\overline{\mathcal{N}}$.

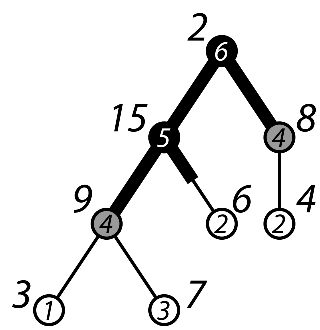

Figure 8: A painted $\mathcal{B}$-forest and a suitable labeling of its nodes, which gives $\pi' = 3|46|7|8910|15|2$.

First we show that $\mathcal{N} \subseteq \mathcal{N}_{\pi'}(\mathcal{B})$. As before, let $j(B) = \max\{j \,:\, B \cap \pi'_j \neq \emptyset\}$. Indeed, if we had $N \in \mathcal{N} \backslash \mathcal{N}_{\pi'}(\mathcal{B})$, we would have $N \subsetneq B \in \mathcal{B}$ with $j(N) = j(B) = j$. Consider the maximal sets $N_1, \ldots, N_k$ of $\mathcal{N}$ that B intersects. They must all have $j(N_i) \leq j$. Since the numbers on the nodes increase strictly up the forest, N must be one of the N_is, and it cannot be the only one. By property (B1) of building sets we conclude that $B \cup N_1 \cup \cdots N_k = N \cup N_1 \cup \cdots N_k \in \mathcal{B}$, which contradicts property (N2) of nested sets.

Now we show that $\mathcal{N}_{\pi'}(\mathcal{B}) \subseteq \mathcal{N}$. Assume we had $B \in \mathcal{N}_{\pi'}(\mathcal{B}) \backslash \mathcal{N}$. Consider the minimal $N \in \mathcal{N}$ containing B. Since $j(B) < j(N)$, B cannot intersect $N_{\pi'}$. Let $N_1, \ldots, N_k$ be the maximal sets in $\mathcal{N}$ that B intersects. They are all strict subsets of N, and there are at least two of them by the minimality of N. Then by (B1) we have $B \cup N_1 \cup \cdots N_k = N_1 \cup \cdots N_k \in \mathcal{B}$, which again contradicts (N2).

We conclude that $\mathcal{N} = \mathcal{N}_{\pi'}(\mathcal{B})$. From the construction of π' we see that block k of π consists of $n+1$ and the union of the sets in $\mathcal{N}^0$, so we also have $\overline{\mathcal{N}} = \overline{\mathcal{N}}_{\pi'}(\mathcal{B})$ as desired.

Finally, we check that this bijection between faces of $\mathcal{JB}$ and painted $\mathcal{B}$-forests is order-reversing. Let F_1 be a face given by a painted $\mathcal{B}$-forest $\mathcal{N}_1$ and let π_1 be a finest partition of $[n+1]$ realizing it, so $F_1 = P_{\pi_1}$. Consider a face F_2 covering F_1; say it corresponds to tree $\mathcal{N}_2$. We can write $F_2 = P_{\pi_2}$ for a partition π_2 obtained from π_1 by merging two parts. If both parts precede (or both succeed) $n+1$ in π_1, then we are contracting a WW edge (or a BB edge) to get from $\mathcal{N}_1$ to $\mathcal{N}_2$. If $n+1$ is its own block in π_1, and it is being merged with a block preceding it (or succeeding it), then we are turning one or more white (or black) vertices into grey vertices. If $n+1$ is not its own block, and it is being merged with a block preceding it (or succeeding it), then we are contracting a GW edge (or contracting a BG bunch.) Therefore $\mathcal{N}_1$ covers $\mathcal{N}_2$. The converse follows by a similar and easier argument. □

Corollary 4.10. *The lifting of the nestohedron $\mathcal{KB}$ is the nestomultiplihedron $\mathcal{JB}$.*

Proof. In light of Proposition 2.5 and Theorem 4.9, this follows from the fact that the polytopes $\sum_{B\in\mathcal{B}} \Delta_B + \sum_{B\in\mathcal{B}} \Delta_{B\cup\{n+1\}}$ and $q\sum_{B\in\mathcal{B}} \Delta_B + (1-q)\sum_{B\in\mathcal{B}} \Delta_{B\cup\{n+1\}}$ have the same combinatorial type. □

Remark 4.11. In [7], Devadoss and Forcey asked for a nice Minkowski decomposition of the *graph multiplihedron $\mathcal{K}G$*. By definition, $\mathcal{K}G$ is combinatorially isomorphic to the nestomultiplihedron for the building set $\mathcal{B}(G)$ of the graph G. Therefore Theorem 4.9 offers an answer to their question.

5 π-liftings and volumes.

We will now modify the lifting operation and define, for each ordered partition π of $[n]$ and $0 \le q \le 1$, the (π, q)-lifting $P^\pi(q)$. This construction is useful in that it subdivides the polytope $P(q)$ into pieces whose volumes are easier to compute; *i.e.*

$$P(q) = \bigcup_{\pi\in\mathcal{P}^n} P^\pi(q), \qquad \text{int } P^{\pi_1}(q) \cap \text{int } P^{\pi_2}(q) = \emptyset \ \text{ for } \pi_1 \ne \pi_2,$$

so

$$\text{Vol}_n(P(q)) = \sum_{\pi\in\mathcal{P}^n} \text{Vol}_n(P^\pi(q)).$$

We will see that $\text{Vol}_n(P^\pi(q))$ is an interesting polynomial in q, which we will explore in greater depth in Part 2.

For the sake of visualization and the cleanliness of formulas, in this section we will treat $P(q)$ as a full-dimensional polytope in $\mathbb{R}^n$ via projection onto the hyperplane $x_{n+1} = 0$, rather than as a polytope of codimension 1 in $\mathbb{R}^{n+1}$. Thus if $P = P_n(\{z_I\})$ then it follows from Proposition 2.3 that $P(q)$ will have hyperplane description

$$P(q) = \left\{ x \in \mathbb{R}^n : qz_I \le \sum_{i\in I} x_i \le z_{[n]} - z_{[n]\setminus I} \text{ for all } I \subseteq [n] \right\}.$$

Definition 5.1. *Let P be a generalized permutahedron in $\mathbb{R}^n$. Let $\pi = B_1|\cdots|B_k$ be an ordered partition of $[n]$ and let $0 \le q \le 1$. Let P_π be the face of P that maximizes a linear functional of type π. For $i = 0, \ldots, k$ construct a modified copy P_π^i of P_π by* *applying a factor of q to the coordinates of the vertices of P_π whose indices belong to the first i blocks of π, $B_1 \cup \cdots \cup B_i$. The convex hull of all of these modified copies of P_π is the* (π, q)-lifting *of P. We denote it as $P^\pi(q)$, and sometimes we simply call it the* π-lifting *of P.*

Note that each ordered partition π corresponds to a different π-lifting $P^\pi(q)$. Even if $P_\pi = P_\mu$, the π-liftings $P^\pi(q)$ and $P^\mu(q)$ will be distinct for $\pi \neq \mu$.

Example 5.2. Consider the associahedron $\mathcal{K}(4)$. Since $\mathcal{K}(4)_{1|3|2}$ is the point $(1,4,1)$, the π-lifting

$$\mathcal{K}(4)^{1|3|2}(q) = \mathrm{conv}\{(1,4,1), (q,4,1), (q,4,q), (q,4q,q)\}.$$

This and other π-liftings are pictured in Figure 9.

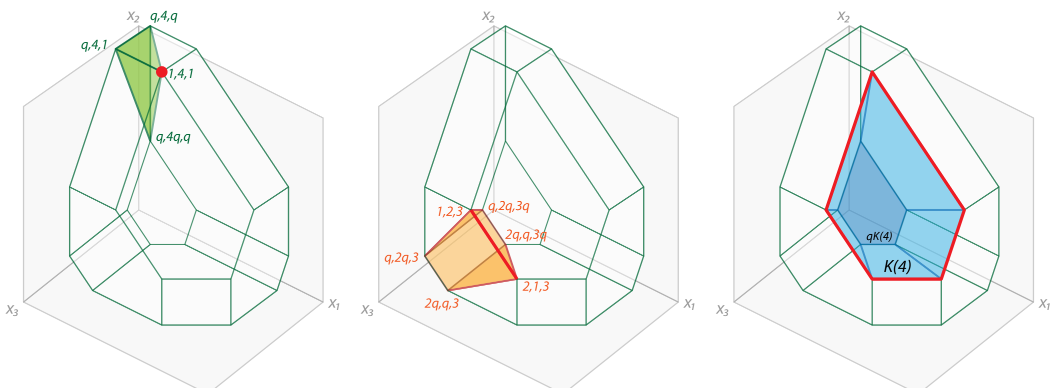

Figure 9: Three π-liftings of the associahedron $\mathcal{K}(4)$: $\mathcal{K}(4)^{1|3|2}(q)$, $\mathcal{K}(4)^{12|3}(q)$, and $\mathcal{K}(4)^{123}(q)$. The bold regions represent the faces $\mathcal{K}(4)_\pi$.

Definition 5.3. *For a subset $I \subseteq [n]$ define $x_I := \sum_{i \in I} x_i$. For a generalized permutahedron $P = P_n(\{z_I\})$ and an ordered partition $\pi = B_1|\cdots|B_k$ define*

$$z_\pi^{B_i} := z_{B_1 \cup \cdots \cup B_i} - z_{B_1 \cup \cdots \cup B_{i-1}}$$

For a minimal refinement $\pi' = B_1|\cdots|B_{i-1}|C_i|D_i|B_{i+1}|\cdots|B_k$, where $B_i = C_i \sqcup D_i$ is a disjoint union, we have

$$z_{\pi'}^{C_i} := z_{B_1 \cup \cdots \cup B_{i-1} \cup C_i} - z_{B_1 \cup \cdots \cup B_{i-1}} \text{ and}$$

$$z_{\pi'}^{D_i} := z_{B_1 \cup \cdots \cup B_i} - z_{B_1 \cup \cdots \cup B_{i-1} \cup C_i}.$$

Proposition 5.4. *For a generalized permutahedron $P = P_n(\{z_I\})$ and an ordered partition $\pi = B_1|\cdots|B_k$ the π-lifting $P^\pi(q)$ has the hyperplane description:*

$$q \;\leq\; \frac{x_{B_1}}{z^{B_1}} \;\leq\; \cdots \;\leq\; \frac{x_{B_k}}{z^{B_k}} \;\leq\; 1, \text{ and} \tag{2}$$

$$\frac{x_{C_i}}{z_{\pi'}^{C_i}} \;\geq\; \frac{x_{D_i}}{z_{\pi'}^{D_i}} \qquad \text{for all } i \text{ and all disjoint decompositions } B_i = C_i \sqcup D_i. \tag{3}$$

For reasons to become clear later, we call the inequalities of the first type the *simplicial inequalities* of $P^\pi(q)$, and those of the second type the *facial inequalities*. Since $x_{C_i} + x_{D_i} = x_{B_i}$ and $z_{\pi'}^{C_i} + z_{\pi'}^{D_i} = z_\pi^{B_i}$, the facial inequalities can be rewritten as

$$\frac{x_{C_i}}{z_{\pi'}^{C_i}} \geq \frac{x_{B_i}}{z_\pi^{B_i}} \quad \text{or equivalently as} \quad \frac{x_{D_i}}{z_{\pi'}^{D_i}} \leq \frac{x_{B_i}}{z_\pi^{B_i}}.$$

Proof of Proposition 5.4. First we claim that any vertex (and hence any point) of $P^\pi(q)$ satisfies the given inequalities. The face P_π consists of the points x in P that satisfy $x_{B_i} = z_\pi^{B_i}$ for $i = 1, \ldots, k$. For any vertex v of P_π^j, $\frac{v_{B_i}}{z_\pi^{B_i}}$ equals q if $i \leq j$ and 1 if $i > j$, so v satisfies the simplicial inequalities (2). Now, for any vertex v of P_π we have $v_{D_i} + (z_{[n]} - z_{B_1 \cup \cdots \cup B_i}) = v_{D_i} + v_{B_{i+1}} + \cdots + v_{B_k} = v_{D_i \cup B_{i+1} \cup \cdots \cup B_k} \leq z_{[n]} - z_{B_1 \cup \cdots \cup B_{i-1} \cup C_i}$ so $\frac{v_{D_i}}{z_{\pi'}^{D_i}} \leq 1 = \frac{v_{B_i}}{z_\pi^{B_i}}$. So all vertices of P_π, and therefore those of P_π^j satisfy the facial inequalities (3) as well. The claim follows.

Conversely, given a point x which satisfies the given inequalities, we show that $x \in P^\pi(q)$. Define $p \in \mathbb{R}^n$ by

$$p|_{B_i} = x|_{B_i} \cdot \frac{z_\pi^{B_i}}{x_{B_i}} \qquad \text{for } i = 1, \ldots, k.$$

We have $p_{B_i} = z_\pi^{B_i}$ for all i. Let p^i be the point obtained from p by multiplying the entries in $B_1 \cup \cdots \cup B_i$ by q. We will show that x is a convex combination of $p^0, p^1, \ldots, p^k$, and that these $k+1$ points are in $P^\pi(q)$. This will imply that $x \in P^\pi(q)$.

To show the first claim, we write $a_i = \left(\frac{x_{B_i}}{z_\pi^{B_i}}\right)/(1-q)$, and compute

$$\begin{aligned} x &= \sum_{i=1}^k x|_{B_i} = \sum_{i=1}^k p|_{B_i} \cdot \frac{x_{B_i}}{z_\pi^{B_i}} = \sum_{i=1}^k \frac{p^{i-1} - p^i}{1-q} \frac{x_{B_i}}{z_\pi^{B_i}} \\ &= \sum_{i=1}^k (p^{i-1} - p^i) \cdot a_i = p^0 a_1 + \sum_{i=1}^{k-1} p^i (a_{i+1} - a_i) - p^k a_k \\ &= p^0 \left(a_1 - \frac{q}{1-q}\right) + \sum_{i=1}^{k-1} p^i \left(a_{i+1} - a_i\right) + p^k \left(\frac{1}{1-q} - a_k\right), \end{aligned}$$

where the coefficients are non-negative by assumption, and add up to 1, as desired.

Now we prove that $p \in P_\pi$, which will imply that $p^i \in P_\pi^i \subset P^\pi(q)$ for all i. By definition p satisfies all the equalities $x_{B_i} = z_\pi^{B_i}$ for $i = 1, \ldots, k$ that hold in the face P_π. Now let us check that it satisfies all inequalities as well. We need to check that $p_C \geq z_C$ for all $C \subseteq [n]$. Write $C = C_1 \cup \cdots \cup C_k$ for $C_i \subseteq B_i$, so $p_C = p_{C_1} + \cdots + p_{C_k}$. Applying the facial inequalities, we have

$$p_{C_i} = x_{C_i} \cdot \frac{z_\pi^{B_i}}{x_{B_i}} \geq z_{\pi'}^{C_i} = z_{B_1 \cup \cdots \cup B_{i-1} \cup C_i} - z_{B_1 \cup \cdots \cup B_{i-1}}$$

The supermodularity of z then gives

$$p_{C_i} \geq z_{C_1 \cup \cdots \cup C_{i-1} \cup C_i} - z_{C_1 \cup \cdots \cup C_{i-1}}$$

which implies that $p_C \geq z_{C_1 \cup \cdots \cup C_k} = z_C$ as desired. □

Corollary 5.5. *The π-lifting $P^\pi(q)$ can be decomposed into the Minkowski sum*

$$P^\pi(q) = qP_\pi + (1-q)P^\pi(0).$$

Proof. Proposition 5.4 tells us the facet directions of the three polytopes involved. The result then follows from the fact that hyperplane parameters are additive under Minkowski sums. □

Now we show that the different π-liftings $P^\pi(q)$ fit together to subdivide $P(q)$, as illustrated in Figure 10.

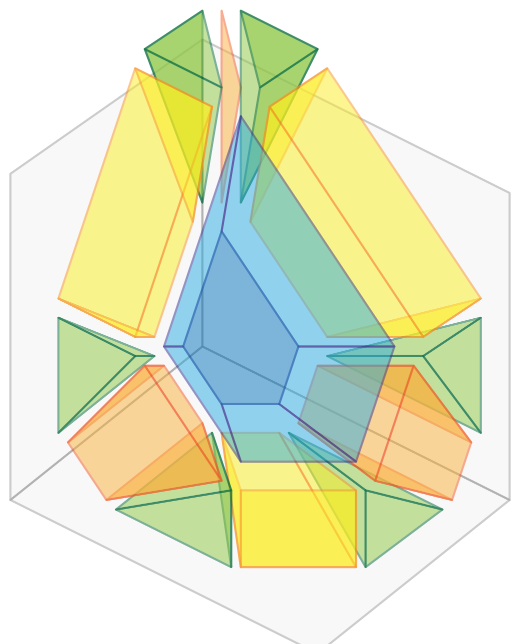

Figure 10: The subdivision of a lifted generalized permutahedron.

Proposition 5.6. *The set of π-liftings $\{P^\pi(q) : \pi$ an ordered partition of $[n]\}$ forms a subdivision of the q-lifted polytope $P(q)$.*

Proof. Let $\pi = B_1 | \cdots | B_k$ be an ordered partition and let $A_i = B_1 \cup \cdots \cup B_i$. Recall that we have assumed that P has been translated to sit in the interior of the positive orthant of $\mathbb{R}^n$. This means that every $x \in P$ will have all strictly positive coordinates, and $z_I < z_J$ for $I \subsetneq J$. We will now reinterpret the inequality description parameters of $P^\pi(q)$ in terms of slopes. For a point $x \in \mathbb{R}^n$ let $v_I =$

$(z_I, x_I) \in \mathbb{R}^2$, where $x_I = \sum_{i\in I} x_i$ as above. For $x \in P^\pi(q)$ the term $\frac{x_{B_i}}{z^{B_i}} = \frac{x_{A_i}-x_{A_{i-1}}}{z_{A_i}-z_{A_{i-1}}}$ is the slope of the segment joining $v_{A_{i-1}}$ and v_{A_i}. Thus the simplicial inequalities in Proposition 5.4 can be interpreted as stating that, starting at the origin $v_{A_0} = v_\emptyset$, the points $v_{A_0}, v_{A_1}, v_{A_2}, \ldots, v_{A_k}$ form a broken line of ascending slopes. Similarly, the facial inequalities state that all points v_C with $A_{i-1} \subset C \subset A_i$ lie on or above the segment connecting $v_{A_{i-1}}$ and v_{A_i}.

Now given a point $x \in P(q)$ construct a partition π as follows. Draw the 2^n points v_I, take the convex hull to create a polygon Q, and look at the "lower hull" of Q, which consists of the edges Q that maximize a linear functional whose second component is nonpositive. This will form a broken line of ascending slopes connecting vertices $v_{A_0}, v_{A_1}, \ldots, v_{A_k}$. Because the x_i are strictly positive we know v_{A_0} will be the origin, and because of the increasing condition on the z_I we know $A_k = [n]$. Now we claim that $A_{i-1} \subset A_i$ for all i.

Suppose by way of contradiction that, ordered from left to right, v_A and v_B are consecutive vertices in the lower hull of Q, but that $A \not\subset B$. By the increasing condition on the z_I we have $z_{A\cap B} < z_A < z_B < z_{A\cup B}$. Moreover, because v_A and v_B are vertices of the lower hull of Q we know that the slope of the line segment connecting $v_{A\cap B}$ and v_A is strictly less than the slope of the segment between v_A and v_B, which is in turn strictly less than the slope of the segment between v_B and $v_{A\cup B}$. Thus

$$\frac{x_A - x_{A\cap B}}{z_A - z_{A\cap B}} < \frac{x_{A\cup B} - x_B}{z_{A\cup B} - z_B}.$$

Notice that the numerators on both sides of this inequality are equal and positive, so we may rearrange terms to get

$$z_A + z_B > z_{A\cup B} + z_{A\cap B},$$

which violates the submodularity condition on the z_I. This is a contradiction.

Now we may let $\pi = B_1|\cdots|B_k$ where $B_i = A_i \setminus A_{i-1}$. By construction x satisfies the simplicial inequalities of $P^\pi(q)$, and by the increasing property of the z_I, x satisfies the facial inequalities as well. Therefore $x \in P^\pi(q)$.

Finally, note that if x is generic then the partition π is uniquely determined by the construction above. Therefore $P^{\pi_1}(q)$ and $P^{\pi_2}(q)$ have disjoint interiors for $\pi_1 \neq \pi_2$. □

Corollary 5.7. *The volume of the q-lifted polytope $P(q)$ is given by*

$$\mathrm{Vol}_n(P(q)) = \sum_{\pi\in\mathcal{P}^n} z_\pi \mathrm{Vol}_{n-k}(P_\pi)\, g_{c(\pi)}(q).$$

Motivated by this result, we now investigate the π-liftings $P^\pi(q)$ and their volumes in detail.

Proposition 5.8. *For $0 < q < 1$, the π-lifting $P^\pi(q)$ is combinatorially isomorphic to $\Delta_k \times P_\pi$.*

Proof. We prove the following stronger statement:

Suppose that, in the inequality description of $P^\pi(q)$ in Proposition 5.4, we keep all the facial inequalities (3) and t of the simplicial inequalities (2), and set the rest to be equalities. Then the resulting face Q of $P^\pi(q)$ is combinatorially isomorphic to $\Delta_{t-1} \times P_\pi$.

Notice that $t \geq 1$ since $q < 1$. First we prove the statement for $t = 1$. Since P is a generalized permutahedron, the π-maximal face $P_\pi = P_1 \times \cdots \times P_k$ for some polytopes $P_1 \subset \mathbb{R}^{B_1}, \ldots, P_k \subset \mathbb{R}^{B_k}$. If we set all but the ith facial inequality (3) to equalities, one easily checks that $Q = qP_1 \times \cdots \times qP_{i-1} \times P_i \times \cdots \times P_k$. Since $q > 0$, Q is combinatorially isomorphic to P_π.

Now we proceed by induction on $s := \dim P^\pi + t$. The base case $s = 1$ follows from the previous paragraph. Now consider a face Q with $\dim P^\pi + t = s$. The facets of Q are the following:

Simplicial: If $t = 1$ then we already showed that Q is isomorphic to $\Delta_0 \times P_\pi$. If $t \geq 1$ and we set any one of the remaining t simplicial inequalities into an equality, the inductive hypothesis assures us that the result is isomorphic to $\Delta_{t-2} \times P_\pi$.

Facial: Consider a facet of Q given by an equation $x_{C_i}/z_{\pi'}^{C_i} = x_{D_i}/z_{\pi'}^{D_i}$. A vertex v of $P \subset P^\pi(q)$ is on this facet if and only if $v \in P^{\pi'}(q)$. In turn, a "q-lifting" of v are on this facet if and only if v is, since the lifting process applies a factor of q to v_{C_i} if and only if it applies it to v_{D_i}. Therefore this facet equals $P^{\pi'}(q)$, and is isomorphic to $\Delta_{t-1} \times P_{\pi'}$ by the inductive hypothesis.

From this it follows that Q is combinatorially isomorphic to $\Delta_{t-1} \times P_\pi$, as we wished to show. □

Theorem 5.9. *Let P be a generalized permutahedron in $\mathbb{R}^n$. Let $\pi = B_1|\cdots|B_k$ be an ordered partition of $[n]$. Then the volume of the π-lifting $P^\pi(q)$ is a polynomial in q given by*

$$\mathrm{Vol}_n(P^\pi(q)) = z_\pi \mathrm{Vol}_{n-k}(P_\pi) \int_q^1 \int_q^{t_k} \cdots \int_q^{t_2} t_1^{|B_1|-1} \cdots t_k^{|B_k|-1} dt_1 \cdots dt_k,$$

where $z_\pi = z_\pi^{B_1} \cdots z_\pi^{B_k}$.

Proof. Consider the projection

$$\begin{array}{rcl} f : \mathbb{R}^n & \to & \mathbb{R}^k \\ x & \mapsto & (x_{B_1}, \ldots, x_{B_k}) \end{array}$$

which maps $P^\pi(q)$ onto the k-simplex $\Delta := \{y \in \mathbb{R}^k : q \leq y_1 \leq \cdots \leq y_k \leq 1\}$. Because P is a generalized permutahedron, the π-maximal face $P_\pi = P_1 \times \cdots \times P_k$ for some polytopes $P_1 \subset \mathbb{R}^{B_1}, \ldots, P_k \subset \mathbb{R}^{B_k}$. One easily checks that

$$f^{-1}(p) = \left(\frac{p_{B_1}}{z_\pi^{B_1}} \cdot P_1\right) \times \cdots \times \left(\frac{p_{B_1}}{z_\pi^{B_k}} \cdot P_1\right)$$

for any $p \in \Delta$. Therefore this fiber is combinatorially isomorphic to P_π and

$$\mathrm{Vol}_{n-k}(f^{-1}(p)) = \left(\frac{p_{B_1}}{z_\pi^{B_1}}\right)^{|B_1|-1} \cdots \left(\frac{p_{B_k}}{z_\pi^{B_k}}\right)^{|B_k|-1} \mathrm{Vol}_{n-k}(P_\pi).$$

The result now follows by integrating over the simplex Δ, using the substitution $t_i := \frac{p_{B_i}}{z^{B_i}}$. $\square$

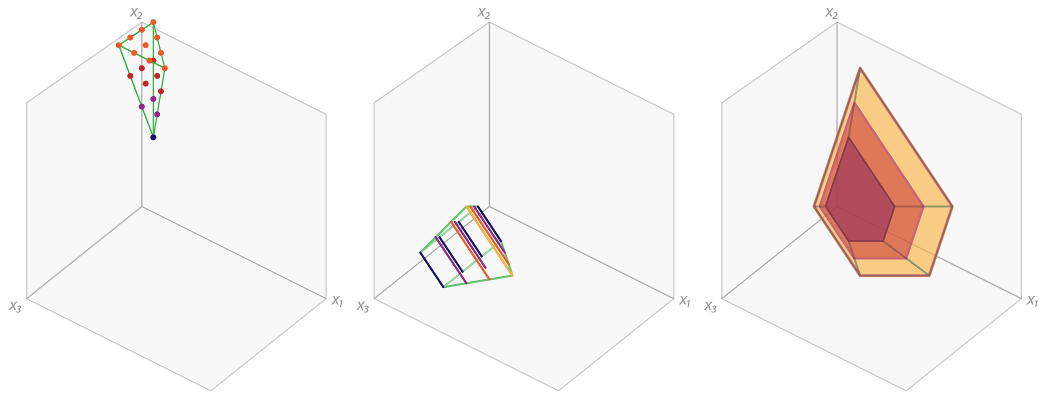

Figure 11: The π-liftings of the associahedron $\mathcal{K}(4)$, $\mathcal{K}(4)^{1|3|2}(q)$, $\mathcal{K}(4)^{12|3}(q)$, and $\mathcal{K}(4)^{123}(q)$ of Figure 9, together with some of the fibers that we are integrating to obtain their volume. The fibers are points, segments, and pentagons, respectively.

Observe that the above integral evaluates to a polynomial in q and depends only on the sizes of the blocks of π. The sequence of these block sizes can be thought of as a composition $c(\pi)$ of the integer n. Let us call this polynomial $g_{c(\pi)}(q)$. This polynomial will be the subject of study of Part 2.

PART 2. COMPOSITION POLYNOMIALS.

In Section 6, motivated by the geometric considerations of Part 1, we introduce the *composition polynomial* $g_c(q)$ of an ordered composition $c = (c_1, \dots, c_k)$ of n and the *reduced composition polynomial* $f_c(q) = (1-q)^{-k} g_c(q)$. We present our main results, Theorems 6.3 – 6.7

In Section 7 we derive an explicit formula (Theorem 6.3) and various properties (Theorem 6.4) of composition polynomials, and we prove the positivity of $f_c(q)$. (Theorem 6.5) In Section 8 we show that composition polynomials arise very naturally in the polynomial interpolation of the exponential function $h(x) = q^x$. (Theorem 6.6) In Section 9 we establish a connection between composition polynomials and Stanley's order polytopes. (Theorem 6.7) We use this to interpret $g_c(q)$ as a generating function for counting linear extensions of a poset P_c. We conclude by suggesting some questions in Section 10.

6 Composition polynomials.

Definition 6.1. *A* composition $c = (c_1, \dots, c_k)$ *is a finite ordered tuple of positive integers. We call the c_i the* parts *of c, and the sum $c_1 + \cdots + c_k$ the* size *of c. If $c = (c_1, \dots, c_k)$ has size n, we say that c is a composition of n into k parts. The* reverse *of the composition c is defined as $\bar{c} = (c_k, \dots, c_1)$.*

Definition 6.2. *For a composition $c = (c_1, \dots, c_k)$ we write $\mathbf{t^{c-1}} := t_1^{c_1-1} \cdots t_k^{c_k-1}$, where $t = (t_1, \dots, t_k)$. The* composition polynomial $g_c(q)$ *is*

$$g_c(q) := \int_q^1 \int_q^{t_k} \cdots \int_q^{t_2} \mathbf{t^{c-1}} dt_1 \cdots dt_k.$$

The reduced composition polynomial *of c is $f_c(q) = g_c(q)/(1-q)^k$.*

It is clear that $g_c(q)$ is indeed a polynomial in q of degree n. It is less clear that $f_c(q)$ is also a polynomial, but we will prove it in Theorem 6.4. Below are some examples of composition polynomials which hint at some of their general properties.

- $g_{(1,1,1,1)}(q) = \frac{1}{24}(1-q)^4$.
- $g_{(2,2,2,2)}(q) = \frac{1}{384}(1-q)^4(1+q)^4$.
- $g_{(1,2,2)}(q) = \frac{1}{120}(1-q)^3(8+9q+3q^2)$.
- $g_{(2,2,1)}(q) = \frac{1}{120}(1-q)^3(3+9q+8q^2)$.
- $g_{(5,3)}(q) = \frac{1}{120}(1-q)^2(5+10q+15q^2+12q^3+9q^4+6q^5+3q^6)$.
- $g_{(a,b)}(q) = \frac{1}{ab(a+b)}(1-q)^2(b+2bq+\cdots+(a-2)bq^{a-3}+(a-1)bq^{a-2}+abq^{a-1}+$ $+a(b-1)q^a+a(b-2)q^{a+1}+\cdots+2aq^{a+b-3}+aq^{a+b-2})$.

Our main results in Part 2 are the following:

Theorem 6.3. *If $\beta_i = c_1 + \cdots + c_i$ for $0 \le i \le k$, we have*

$$g_c(q) = \sum_{i=0}^{k} \frac{q^{\beta_i}}{\prod_{j\neq i}(\beta_j - \beta_i)}.$$

Theorem 6.4. *Let $c = (c_1, \ldots, c_k)$ be a composition of n. Then:*

1. $g_{\bar{c}}(q) = q^n g_c(1/q)$.
2. $g_{mc}(q) = \frac{1}{m^k} g_c(q^m)$ *for any positive integer m.*
3. $g_c(q) = (1-q)^k f_c(q)$ *for a polynomial $f_c(q)$ of degree $n-k$ with $f_c(1) \neq 0$.*
4. $f_c(1) = 1/k!$.

Theorem 6.5. *The coefficients of the reduced composition polynomial $f_c(q)$ are positive.*

Theorem 6.6. *Let $c = (c_1, \ldots, c_k)$ be a composition and let $\beta_i = c_1 + \cdots + c_i$ for $i = 0, \ldots, k$. Let $h(x) = a_0 + a_1 x + \cdots + a_k x^k$ be the polynomial of smallest degree that passes through the $k+1$ points (β_i, q^{β_i}). Here the coefficients a_i are functions of q. Then $a_k = (-1)^k g_c(q)$.*

Theorem 6.7. *There is a poset P_c and an element $p \in P_c$ such that the volume of a slice of the order polytope $\mathcal{O}(P_c)$ in the x_p direction is*

$$\mathrm{Vol}\,(\mathcal{O}(P_c) \cap (x_p = q)) = \frac{g_c(q)}{(c_1-1)! \cdots (c_k-1)!}.$$

7 Recursive and explicit formulas

Definition 7.1. *Define the* truncated compositions $c^L := (c_2, \ldots, c_k)$ *and* $c^R := (c_1, \ldots, c_{k-1})$. *For $m \in \{1, \ldots, k-1\}$ we define the* merged composition c^m *as the composition formed by combining the parts c_m and c_{m+1} into a single part:*

$$c^m := (c_1, \ldots, c_{m-1}, c_m + c_{m+1}, c_{m+2}, \ldots, c_k).$$

Lemma 7.2. *For a composition $c = (c_1, \ldots, c_k)$ of n, the composition polynomial $g_c(q)$ satisfies the recursion:*

$$g_c(q) = \frac{1}{c_1} g_{c^1}(q) - \frac{q^{c_1}}{c_1} g_{c^L}(q).$$

Proof. We have:

$$\begin{aligned} g_c(q) &= \int_q^1 \int_q^{t_k} \cdots \int_q^{t_2} t_1^{c_1-1} \cdots t_k^{c_k-1} dt_1 \cdots dt_k \\ &= \frac{1}{c_1} \int_q^1 \int_q^{t_k} \cdots \int_q^{t_3} t_2^{c_2-1} \cdots t_k^{c_k-1} (t_2^{c_1} - q^{c_1}) dt_2 \cdots dt_k \\ &= \frac{1}{c_1} g_{(c_1+c_2, c_3, \ldots, c_k)}(q) - \frac{q^{c_1}}{c_1} g_{(c_2, c_3, \ldots, c_k)}(q) \end{aligned}$$

as we wished to show. □

Consider the sequence of partial sums $0 = \beta_0 < \cdots < \beta_k = n$ by $\beta_i = c_1 + \cdots + c_i$ for $i = 1, \ldots, k$. Let (β) denote the Vandermonde matrix

$$(\beta) = \begin{pmatrix} 1 & \beta_0 & \cdots & \beta_0^k \\ \vdots & \vdots & & \vdots \\ 1 & \beta_k & \cdots & \beta_k^k \end{pmatrix}.$$

We will index the rows and columns of this matrix from 0 to k. Recall that

$$\det(\beta) = \prod_{0 \leq i < j \leq k} (\beta_j - \beta_i).$$

For $0 \leq i \leq k$ let

$$[\beta_i] := (-1)^i \prod_{j \neq i} (\beta_j - \beta_i), \qquad [\hat{\beta}_i] := \det(\beta)/[\beta_i].$$

Notice that $[\hat{\beta}_i]$ is the unsigned minor of (β) obtained by removing row i and column k. Moreover, $[\hat{\beta}_i]$ is itself a Vandermonde determinant. We are ready to prove our explicit formula for composition polynomials, which we rewrite as:

$$g_c(q) = \sum_{i=0}^{k} (-1)^i \frac{q^{\beta_i}}{[\beta_i]}.$$

Proof of Theorem 6.3. Define $[\beta_i^R]$ analogously to $[\beta_i]$ for the truncated composition $c^R = (c_1, \ldots, c_{k-1})$. Proceed by induction on k. If $k = 1$ then

$$\int_q^1 t_1^{c_1 - 1} dt_1 = \frac{1}{c_1} - \frac{q^{c_1}}{c_1} = \frac{q^{\beta_0}}{[\beta_0]} - \frac{q^{\beta_1}}{[\beta_1]}.$$

Now assume that the formula holds up to $k - 1$. Then

$$g_{c^R}(q) = \int_q^1 \cdots \int_q^{t_2} \mathbf{t}^{\mathbf{c^R - 1}} dt_1 \cdots dt_{k-1} = \sum_{i=0}^{k-1} (-1)^i \frac{q^{\beta_i}}{[\beta_i^R]}.$$

Changing the upper bound of the outer integral produces

$$\int_q^{t_k} \cdots \int_q^{t_2} \mathbf{t}^{\mathbf{c^R - 1}} dt_1 \cdots dt_{k-1} = \sum_{i=0}^{k-1} (-1)^i \frac{q^{\beta_i} t_k^{\beta_{k-1} - \beta_i}}{[\beta_i^R]}.$$

This follows from the observation that this integral must evaluate to a homogeneous polynomial in t_k and q of total degree $c_1 + \cdots + c_{k-1} = \beta_{k-1}$. The original integral we wish to compute becomes

$$\begin{aligned} g_c(q) &= \int_q^1 t_k^{c_k - 1} \sum_{i=0}^{k-1} (-1)^i \frac{q^{\beta_i} t_k^{\beta_{k-1} - \beta_i}}{[\beta_i^R]} dt_k \\ &= \sum_{i=0}^{k-1} (-1)^i q^{\beta_i} \int_q^1 \frac{t_k^{\beta_k - \beta_i - 1}}{[\beta_i^R]} dt_k \\ &= \sum_{i=0}^{k-1} (-1)^i \frac{q^{\beta_i}}{[\beta_i]} - q^{\beta_k} \sum_{i=0}^{k-1} \frac{(-1)^i}{[\beta_i]}. \end{aligned}$$

Now observe that $(\beta)\sum_{i=0}^{k}(-1)^i/[\beta_i] = \sum_{i=0}^{k}(-1)^i[\hat{\beta}_i]$ computes, up to sign, the determinant of the matrix formed by replacing the last column in the Vandermonde matrix (β) with a column of 1s. This determinant is clearly zero, hence $-\sum_{i=0}^{k-1}(-1)^i/[\beta_i] = (-1)^k/[\beta_k]$. This gives us the desired result. □

Corollary 7.3. *Given a composition $c = (c_1, \ldots, c_k)$, the composition polynomials of the associated merged and truncated compositions are given by*

$$g_{c^m}(q) = \sum_{i=0}^{k}(-1)^i \frac{q^{\beta_i}(\beta_m - \beta_i)}{[\beta_i]},$$

$$g_{c^R}(q) = \sum_{i=0}^{k}(-1)^i \frac{q^{\beta_i}(n - \beta_i)}{[\beta_i]}, \text{ and}$$

$$q^{c_1} g_{c^L}(q) = -\sum_{i=0}^{k}(-1)^i \frac{q^{\beta_i}\beta_i}{[\beta_i]}.$$

Proof. For the merged composition c^m, the partial sums β_i^m are given by $\beta_i^m = \beta_i$ for $i < m$, and $\beta_i^m = \beta_{i+1}$ for $i \geq m$. From this observe that $[\beta_i^m] = [\beta_i]/(\beta_m - \beta_i)$ for $i < m$ and $[\beta_i^m] = [\beta_{i+1}]/(\beta_{i+1} - \beta_m)$ for $i \geq m$. Notice that the coefficient of q^{β_m} is zero, as it should be.

For the truncated composition c^R the partial sums β_i^R follow this same pattern. Finally, for the truncation c^L we have $\beta_i^L = \beta_{i+1} - \beta_1$ for $i \geq 1$, and $\beta_0^L = 0$. From this we observe that $[\beta_i^L] = [\beta_{i+1}]/\beta_{i+1}$ for all i. Substituting into Theorem 6.3 yields the desired formulas. □

Now we can write down a stronger recursive formula for $g_c(q)$ that will be the key to our proof of Theorem 6.5.

Corollary 7.4. *Let $c = (c_1, \ldots, c_k)$ be a composition of n into k parts. Let c^m be the merged composition $(c_1, \ldots, c_m + c_{m+1}, \ldots, c_k)$, and let $c^L = (c_2, \ldots, c_k)$ and $c^R = (c_1, \ldots, c_{k-1})$ be the truncated compositions. Then*

$$g_{c^m}(q) = \left(\frac{c_1 + \cdots + c_m}{c_1 + \cdots + c_k}\right) g_{c^R}(q) + \left(\frac{c_{m+1} + \cdots + c_k}{c_1 + \cdots + c_k}\right) q^{c_1} g_{c^L}(q). \tag{4}$$

Proof. This follows immediately from Corollary 7.3. □

It is possible to write down several recursive equations for g_c, but this particular one is significant for several reasons:

- Every non-trivial composition c can be thought of as a merged composition for some m, and the sizes of c^L and c^R are each strictly less than the size of c^m. This means we have actually produced a recursive expression for an arbitrary nontrivial composition polynomial in terms of "smaller" composition polynomials. This will allow us to prove Theorem 6.4 inductively.

- The compositions c^m, c^L, and c^R have the same length, so the polynomials f_c turn out to satisfy exactly the same recursion as g_c by Theorem 6.4.4.

- Since this recursion only has positive terms, we will then obtain a proof of Theorem 6.5, the positivity of f_c.

Proof of Theorems 6.4 and 6.5. Parts 1. and 2. of Theorem 6.4 follow readily from our explicit formula for $g_c(q)$. The partial sums of the reversal $\bar{c}$ are $\bar{\beta}_i = n - \beta_{k-i}$, and $[\bar{\beta}_i] = [\beta_{k-i}]$. The partial sums of mc are $m\beta_i$, and $[m\beta_i] = m^k[\beta_i]$. Substituting these into Theorem 6.3 gives the results.

We prove Theorems 6.4.3, 6.4.4, and 6.5 by induction on the size of c for a fixed k. The base case is $c = (1, \ldots, 1)$, the composition of k into k parts. Theorem 6.3 gives

$$g_{(1,\ldots,1)}(q) = \sum_{i=0}^{k} (-1)^i \frac{q^i}{i!(k-i)!} = \frac{1}{k!}(1-q)^k.$$

from which the claims follow readily.

Now suppose c has size $n > k$. Then some part of c is greater than 1, and we can write c as some merged composition c'^m for some composition c'. By Corollary 7.4,

$$g_c(q) = g_{c'^m}(q) = \frac{\beta'_m}{n} g_{c'^R}(q) + \left(1 - \frac{\beta'_m}{n}\right) q^{c'_1} g_{c'^L}(q).$$

Notice that c'^R and c'^L are compositions of length k and size strictly smaller than c. Therefore by induction we may write

$$\begin{aligned} g_c(q) &= \frac{\beta'_m}{n}(1-q)^k f_{c'^R}(q) + \left(1 - \frac{\beta'_m}{n}\right) q^{c'_1}(1-q)^k f_{c'^L}(q) \\ &= (1-q)^k \left(\frac{\beta'_m}{n} f_{c'^R}(q) + \left(1 - \frac{\beta'_m}{n}\right) q^{c'_1} f_{c'^L}(q) \right) \\ &=: (1-q)^k f_c(q). \end{aligned}$$

where $f_c(q)$ is a polynomial of degree $n-k$. Since $\frac{\beta'_m}{n}$ and $\left(1 - \frac{\beta'_m}{n}\right)$ are positive and they sum to 1, f_c inherits the desired properties from $f_{c'^R}$ and $f_{c'^L}$. □

Further examples seem to suggest that the sequence of coefficients of $f_c(q)$ is unimodal, meaning that the coefficients $f_c(q) = \sum_{i=0}^{n-k} f_i q^i$ satisfy the inequalities $f_1 \leq f_2 \leq \cdots \leq f_{i-1} \leq f_i \geq f_{i+1} \geq \cdots \geq f_{n-k}$ for some i. More strongly, the sequence may even be log-concave, meaning that $f_j^2 \geq f_{j-1}f_{j+1}$ for all j. We have verified both statements for all 335,922 compositions of at most 7 parts and sizes of parts at most 6.

Question 7.5. *Is the sequence of coefficients of $f_c(q)$ always unimodal?*

Since $g_c(q)$ is essentially the volume of a Minkowski sum of two polytopes (Proposition 5.5), one might hope to derive the log-concavity of the f_i from the Aleksandrov-Fenchel inequalities [24, 26]. The "obvious" application of these inequalities does not seem to give the desired result, and the question remains open.

We conclude this section with an explicit formula for the coefficients of $f_c(q)$. Unfortunately, this formula does not seem to explain their unimodality, or even their positivity (Theorem 6.5). Recall the notation

$$\left(\!\!\binom{n}{k}\!\!\right) := \binom{n+k-1}{k-1}$$

for the number of multisets of $[n]$ of size k.

Corollary 7.6. *The reduced composition polynomial* $f_c(q) = \sum_{i=0}^{n-k} f_i q^i$ *has*

$$f_i = \sum_{j:\beta_j \le i} \frac{(-1)^j}{[\beta_j]} \left(\!\!\binom{k}{i-\beta_j}\!\!\right).$$

Proof. We compute

$$\begin{aligned} f_c(q) &= g_c(q)(1+q+q^2+\cdots)^k \\ &= \left(\sum_{j=0}^{k}(-1)^j \frac{q^{\beta_j}}{[\beta_j]}\right)\left(\sum_{i=0}^{\infty}\left(\!\!\binom{k}{i}\!\!\right) q^i\right) \end{aligned}$$

as desired. □

8 Composition polynomials in polynomial interpolation.

Now we prove Theorem 6.6, which shows that composition polynomials have a very natural interpretation in terms of the polynomial interpolation of an exponential function $e(x) = q^x$.

Recall that $h(x)$ is the polynomial of smallest degree which agrees with $e(x) = q^x$ at the points $\beta_i = c_1 + \cdots + c_i$. We wish to show that the leading coefficient of $h(x)$, which is a function of q, in fact equals $(-1)^k g_c(q)$.

Proof of Theorem 6.6. Theorem 6.3 implies that $\det(\beta) g_c(q) = \sum_{i=0}^{k}(-1)^i q^{\beta_i}[\hat{\beta}_i]$, which we rewrite as

$$\det(\beta)\cdot g_c(q) = (-1)^k \det\begin{pmatrix} 1 & \beta_0 & \cdots & \beta_0^{k-1} & q^{\beta_0} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 1 & \beta_k & \cdots & \beta_k^{k-1} & q^{\beta_k}\end{pmatrix}. \qquad (5)$$

Now notice that this is, up to sign, precisely what we obtain when we use Cramer's rule to solve for a_k in the system of linear equations

$$\begin{pmatrix} 1 & \beta_0 & \cdots & \beta_0^{k} \\ \vdots & \vdots & \dots & \vdots \\ 1 & \beta_k & \cdots & \beta_k^{k}\end{pmatrix}\begin{pmatrix} a_0 \\ \vdots \\ a_k\end{pmatrix} = \begin{pmatrix} q^{\beta_0} \\ \vdots \\ q^{\beta_k}\end{pmatrix}.$$

But this system is equivalent to the polynomial interpolation problem under consideration. The desired result follows. □

We can also interpret the individual coefficients of $f_c(q)$ in terms of the polynomial interpolation of a polynomial function which has been "shut off" after $q = i$. Consider the function

$$d(x) = \begin{cases} \left(\!\binom{k}{i-x}\!\right), & \text{if } x \le i \\ 0, & \text{if } x > i \end{cases}$$

Proposition 8.1. *Let $f_c(q) = \sum_{i=0}^{n-k} f_i q^i$. Then $(-1)^k f_i$ is the lead coefficient of the polynomial $p_i(x)$ of smallest degree that passes through the points $(\beta_j, h(\beta_j))$ for $j = 0, 1, \ldots, k$.*

Proof. This follows from a similar argument. □

We can use these results to give non-recursive explanations of parts of Theorem 6.4. We need a simple lemma.

Lemma 8.2. *Let $(\beta)^p$ be the matrix formed from the Vandermonde matrix (β) by replacing the entries β_i^k of the last column of (β) with a polynomial $p(\beta_i)$ of degree $d \le k$ and lead coefficient c. Then*

$$\det((\beta)^p) = \begin{cases} 0 & \textit{if } d < k, \\ c \cdot \textit{det}(\beta) & \textit{if } d = k. \end{cases}$$

Proof. For $d = k$ we simply observe that $(\beta)^p$ can be obtained from (β) via elementary column operations. The only such operation that affects the determinant is multiplying the last column of (β) by c. If $d < k$ then the last column of $(\beta)^p$ is a linear combination of the previous columns, and thus the matrix is singular. □

Alternate proof of Theorem 6.4.3 and 6.4.4. Taking the i^{th} derivative of (5) gives

$$\det(\beta) g_c^{(i)}(1) = (-1)^k \det \begin{pmatrix} 1 & \beta_0 & \cdots & \beta_0^{k-1} & \beta_0(\beta_0 - 1)\cdots(\beta_0 - i + 1) \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & \beta_k & \cdots & \beta_k^{k-1} & \beta_k(\beta_k - 1)\cdots(\beta_k - i + 1) \end{pmatrix}.$$

Lemma 8.2 tells us that this equals 0 for $0 \le i \le k-1$ and $(-1)^k \det(\beta)$ for $i = k$. Therefore 1 is a root of order k in $g_c(q)$, and taking the k^{th} derivative of $g_c(q) = (1-q)^k f_c(q)$ we obtain $f_c(1) = \frac{1}{k!}$. □

9 Composition polynomials and order polytopes

Consider the poset P_c consisting of a chain $p_0 < p_1 < \cdots < p_k$ together with a chain of size $c_i - 1$ below p_i for $1 \le i \le k$. The *order polytope* $\mathcal{O}(P_c)$, introduced by Stanley in [25], is the polytope of points $x \in \mathbb{R}^{P_c}$ such that $0 \le x_i \le x_j \le 1$ whenever $i \le j \in P$.

Proposition 9.1. *Let $H \in \mathbb{R}^{P_c}$ be the hyperplane $x_{p_0} = q$. Then*

$$\operatorname{Vol}(\mathcal{O}(P_c) \cap H) = \frac{g_c(q)}{(c_1 - 1)! \cdots (c_k - 1)!}.$$

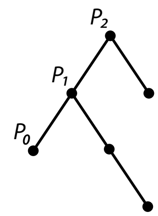

Figure 12: The poset P_{32}.

Proof. For any $0 \le q \le t_1 \le \cdots \le \cdots t_k \le 1$, the intersection of $\mathcal{O}(P_c)$ with $x_{p_0} = q$ and $x_{p_i} = t_i$ for $1 \le i \le k$ is a product of k simplices having volume $\prod_{i=1}^k \frac{t_i^{c_i-1}}{(c_i-1)!}$. Now integrate over all such values. □

Corollary 9.2. *The composition polynomial is given by*

$$g_c(q) = \frac{(c_1-1)!\cdots(c_k-1)!}{n!}\sum_{i=0}^{n} N_{i+1}\binom{n}{i} q^i(1-q)^{n-i}$$

where N_j is the number of linear extensions of P_c such that x_0 has height j. We have $N_j^2 \ge N_{j-1}N_{j+1}$ for $2 \le j \le n$.

Proof. This follows from Stanley's work on order polytopes, namely Proposition 9.1 and (15) of [25]. □

10 Questions and further directions

Our work raises the following questions.

- Find a simple combinatorial interpretation of the coefficients of $f_c(q)$.
- Our proof of Theorem 6.6 does not really explain the connection between the polytopes we study and the fundamental problem of interpolating an exponential function by polynomials. Find a more conceptual proof.
- Settle Question 7.5: Are the coefficients of $f_c(q)$ unimodal? Are they log-concave?

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