Flag arrangements and triangulations of products of simplices.

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Abstract

We investigate the line arrangement that results from intersecting d complete flags in $\mathbb{C}^n$. We give a combinatorial description of the matroid $\mathcal{T}_{n,d}$ that keeps track of the linear dependence relations among these lines.

We prove that the bases of the matroid $\mathcal{T}_{n,3}$ characterize the triangles with holes which can be tiled with unit rhombi. More generally, we provide evidence for a conjectural connection between the matroid $\mathcal{T}_{n,d}$, the triangulations of the product of simplices $\Delta_{n-1} \times \Delta_{d-1}$, and the arrangements of d tropical hyperplanes in tropical $(n-1)$-space.

Our work provides a simple and effective criterion to ensure the vanishing of many Schubert structure constants in the flag manifold, and a new perspective on Billey and Vakil's method for computing the non-vanishing ones.

1 Introduction.

Let $E_\bullet^1, \dots, E_\bullet^d$ be d generically chosen complete flags in $\mathbb{C}^n$. We will work over the field $\mathbb{C}$ of complex numbers, although our results hold over any sufficiently large field. Write

$$E_\bullet^k = \{\{0\} = E_0^k \subset E_1^k \subset \cdots \subset E_n^k = \mathbb{C}^n\},$$

where E_i^k is a vector space of dimension i. Consider the set $\mathbf{E}_{n,d}$ of one-dimensional intersections determined by the flags; that is, all lines of the form $E_{a_1}^1 \cap E_{a_2}^2 \cap \cdots \cap E_{a_d}^d$.

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The initial goal of this paper is to characterize the line arrangements $\mathbb{C}^n$ which arise in this way from d generically chosen complete flags. We will then show an unexpected connection between these line arrangements and an important and ubiquitous family of subdivisions of polytopes: the triangulations of the product of simplices $\Delta_{n-1} \times \Delta_{d-1}$. These triangulations appear naturally in studying the geometry of the product of all minors of a matrix [3], tropical geometry [11], and transportation problems [42]. To finish, we will illustrate some of the consequences that the combinatorics of these line arrangements have on the Schubert calculus of the flag manifold.

The results of the paper are roughly divided into four parts as follows. First of all, Section 2 is devoted to studying the line arrangement determined by the intersections of a generic arrangement of hyperplanes. This will serve as a warmup before we investigate generic arrangements of complete flags, and the results we obtain will be useful in that investigation.

The second part consists of Sections 3, 4 and 5, where we will characterize the line arrangements that arise as intersections of a "matroid-generic" arrangement of d flags in $\mathbb{C}^n$. Section 3 is a short discussion of the combinatorial setup that we will use to encode these geometric objects. In Section 4, we propose a combinatorial definition of a matroid $\mathcal{T}_{n,d}$. In Section 5 we will show that $\mathcal{T}_{n,d}$ is the matroid of the line arrangement of any d flags in $\mathbb{C}^n$ which are generic enough. Finally, we show that these line arrangements are completely characterized combinatorially: any line arrangement in $\mathbb{C}^n$ whose matroid is $\mathcal{T}_{n,d}$ arises as an intersection of d flags.

The third part establishes a surprising connection between these line arrangements and an important class of subdivisions of polytopes. The bases of $\mathcal{T}_{n,3}$ exactly describe the ways of punching n triangular unit holes into the equilateral triangle of size n, so that the resulting holey triangle can be tiled with unit rhombi. A consequence of this is a very explicit geometric representation of $\mathcal{T}_{n,3}$. We show these results in Section 6. We then pursue a higher-dimensional generalization of this result. In Section 7, we suggest that the fine mixed subdivisions of the Minkowski sum $n\Delta_{d-1}$ are an adequate $(d-1)$-dimensional generalization of the rhombus tilings of holey triangles. We give a completely combinatorial description of these subdivisions. Finally, in Section 8, we prove that each fine mixed subdivision of the Minkowski sum $n\Delta_{d-1}$ (or equivalently, each triangulation of the product of simplices $\Delta_{n-1} \times \Delta_{d-1}$) gives rise to a basis of $\mathcal{T}_{n,d}$. We conjecture that every basis of $\mathcal{T}_{n,d}$ arises in this way. In fact, it may be true that every basis of $\mathcal{T}_{n,d}$ arises from a *regular* subdivision or, equivalently, from an arrangement of d tropical hyperplanes in tropical $(n-1)$-space.

The fourth and last part of the paper, Section 9, presents some of the consequences of our work in the Schubert calculus of the flag manifold. We start by recalling Eriksson and Linusson's permutation arrays, and Billey and Vakil's related method for explicitly intersecting Schubert varieties. In Section 9.1 we show how the geometric representation of the matroid $\mathcal{T}_{n,3}$ of Section 6 gives us a new perspective on Billey and Vakil's method for computing the structure constants c_{uvw} of the cohomology ring of the flag variety. Finally, Section 9.2 presents a simple and effective criterion for guaranteeing that many Schubert structure constants are equal to zero.

We conclude with some future directions of research that are suggested by this project.

2 The lines in a generic hyperplane arrangement.

Before thinking about flags, let us start by studying the slightly easier problem of understanding the matroid of lines of a generic arrangement of m hyperplanes in $\mathbb{C}^n$. We will start by presenting, in Proposition 2.1, a combinatorial definition of this matroid $\mathcal{H}_{n,m}$. Theorem 2.2 then shows that this is, indeed, the right matroid. As it turns out, this warmup exercise will play an important role in Section 5.

Throughout this section, we will consider a generic central[1] hyperplane arrangement, consisting of m hyperplanes $H_1, \ldots, H_m$ in $\mathbb{C}^n$. For each subset A of $[m] = \{1, 2, \ldots, m\}$, let

$$H_A = \bigcap_{a \in A} H_a.$$

By genericity,

$$\dim H_A = \begin{cases} n - |A| & \text{if } |A| \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, the set $L_{n,m}$ of one-dimensional intersections of the H_is consists of the $\binom{m}{n-1}$ lines H_A for $|A| = n - 1$.

There are several "combinatorial" dependence relations among the lines in $L_{n,m}$, as follows. Each t-dimensional intersection H_B (where B is an $(n-t)$-subset of $[m]$) contains the lines H_A with $B \subseteq A$. Therefore, in an independent set $H_{A_1}, \ldots, H_{A_k}$ of $L_{n,m}$, we cannot have $t+1$ A_is which contain a fixed $(n-t)$-set B.

[1] A hyperplane arrangement is *central* if all its hyperplanes go through the origin.

At first sight, it seems intuitively clear that, in a generic hyperplane arrangement, these will be the only dependence relations among the lines in $L_{n,m}$. This is not as obvious as it may seem: let us illustrate a situation in $L_{4,5}$ which is surprisingly close to a counterexample to this statement. For simplicity, we will draw the three-dimensional projective picture. Each hyperplane in $\mathbb{C}^4$ will now look two-dimensional, and the lines in the arrangement $L_{4,5}$ will look like points. Denote hyperplanes $H_1, \ldots, H_5$ simply by $1, \ldots, 5$, and an intersection like H_{124} simply by 124.

In Figure 1, we have started by drawing the triangles T and T' with vertices $124, 234, 134$ and $125, 235, 135$, respectively. The three lines connecting the pairs $(124, 125)$, $(234, 235)$ and $(134, 135)$, are the lines $12, 23$, and 13, respectively. They intersect at the point 123, so that the triangles T and T' are perspective with respect to this point.

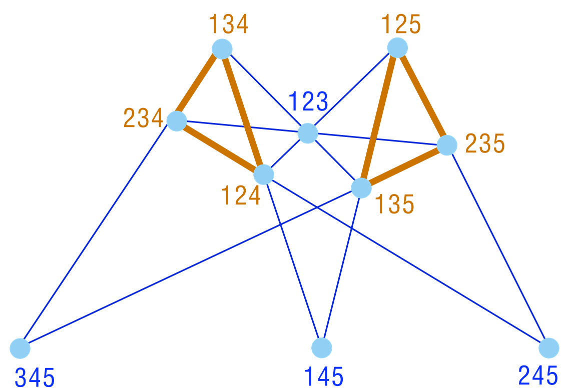

Figure 1: The Desargues configuration in $L_{4,5}$.

Now Desargues' theorem applies, and it predicts an unexpected dependence relation. It tells us that the three points of intersection of the corresponding sides of T and T' are collinear. The lines 14 (which connects 124 and 134) and 15 (which connects 125 and 135) intersect at the point 145. Similarly, 24 and 25 intersect at 245, and 34 and 35 intersect at 345. Desargues' theorem says that the points 145, 245, and 345 are collinear. In principle, this new dependence relation does not seem to be one of our predicted "combinatorial relations". Somewhat surprisingly, it is: it simply states that these three points are on the line 45.

The previous discussion shows that even five generic hyperplanes in $\mathbb{C}^4$ give rise to interesting geometric configurations. In this case, we might consider ourselves fortunate, because the conclusion of Desargues' theorem was also a consequence of our combinatorial relations. However, it is not unreasonable to think that larger arrangements $L_{n,m}$ will contain other configurations, which have nontrivial dependence relations that we may not have predicted.

Having told our readers what they might need to worry about, we now intend to convince them not to worry about it.

First we show that the combinatorial dependence relations in $L_{n,m}$ are consistent, in the sense that they define a matroid. This statement will follow as a consequence of Theorem 2.2. We now give a different proof, which sheds light on the combinatorial structure of the matroid.

Proposition 2.1. *Let $\mathcal{I}$ consist of the collections I of subsets of $[m]$, each containing $n-1$ elements, such that no $t+1$ of the sets in I contain an $(n-t)$-set. In symbols,*

$$\mathcal{I} := \left\{ I \subseteq \binom{[m]}{n-1} \text{ such that for all } S \subseteq I,\ \Big|\bigcap_{A \in S} A\Big| \leq n - |S| \right\}.$$

Then $\mathcal{I}$ is the collection of independent sets of a matroid $\mathcal{H}_{n,m}$.

Proof. A circuit of that matroid would be a minimal collection C of s subsets of $[m]$ of size $n-1$, all of which contain one fixed $(n-s+1)$-set. It suffices to verify the circuit axioms:

(C1) No proper subset of a circuit is a circuit.

(C2) If two circuits C_1 and C_2 have an element x in common, then $C_1 \cup C_2 - x$ contains a circuit.

The first axiom is satisfied trivially. Now consider two circuits C_1 and C_2 containing a common $(n-1)$-set X_1. Let

$$C_1 = \{X_1, \ldots, X_a, Y_1, \ldots, Y_b\}, \qquad C_2 = \{X_1, \ldots, X_a, Z_1, \ldots, Z_c\},$$

where the Y_is and Z_is are all distinct. Write

$$X = \bigcap_{i=1}^{a} X_i, \qquad Y = \bigcap_{i=1}^{b} Y_i, \qquad Z = \bigcap_{i=1}^{c} Z_i.$$

By definition of C_1 and C_2 we have that $|X \cap Y| \geq n - (a+b) + 1$ and $|X \cap Z| \geq n - (a+c) + 1$, and their minimality implies that $|X| \leq n - a$. Therefore

$$\begin{aligned} |X \cap Y \cap Z| &= |X \cap Y| + |X \cap Z| - |(X \cap Y) \cup (X \cap Z)| \\ &\geq |X \cap Y| + |X \cap Z| - |X| \\ &\geq (n-a-b+1) + (n-a-c+1) - (n-a) \\ &= n-a-b-c+2, \end{aligned}$$

and hence

$$|X_2 \cap \cdots \cap X_a \cap Y_1 \cap \cdots \cap Y_b \cap Z_1 \cap \cdots \cap Z_c| \geq n - (a + b + c - 1) + 1.$$

It follows that $C_1 \cup C_2 - X_1$ contains a circuit, as desired.

□

Now we show that this matroid $\mathcal{H}_{n,m}$ is the one determined by the lines in a generic hyperplane arrangement.

Theorem 2.2. *If a central hyperplane arrangement $\mathcal{A} = \{H_1, \ldots, H_m\}$ in $\mathbb{C}^n$ is generic enough, then the matroid of the $\binom{m}{n-1}$ lines H_A is isomorphic to $\mathcal{H}_{n,m}$.*

Proof. We already observed that the one-dimensional intersections of $\mathcal{A}$ satisfy all the dependence relations of $\mathcal{H}_{n,m}$. Now we wish to show that, if $\mathcal{A}$ is generic enough, these are the only relations.

Any hyperplane arrangement can be constructed as follows. Consider the m coordinate hyperplanes in $\mathbb{C}^m$, numbered $J_1, \ldots, J_m$. Pick an n-dimensional subspace V of $\mathbb{C}^m$, and consider the ($(n-1)$-dimensional) arrangement of hyperplanes $H_1 = J_1 \cap V, \ldots, H_m = J_m \cap V$ in V. We will see that, if V is generic enough in the sense of Dilworth truncations, then the arrangement $\{H_1, \ldots, H_m\}$ is generic enough for the conclusion of Theorem 2.2 to hold. We now recall this setup.

Theorem 2.3. *(Brylawski, Dilworth, Mason, [7, 8, 29]) Let L be a set of lines in $\mathbb{C}^r$ whose corresponding matroid is M. Let V be a subspace of $\mathbb{C}^r$ of codimension $k-1$. For each k-flat F spanned by L, let $v_F = F \cap V$.*

1. *If V is generic enough, then each v_F is a line, and the matroid $D_k(M)$ of the lines v_F does not depend on V.*
2. *The circuits of $D_k(M)$ are the minimal sets $\{v_{F_1}, \ldots, v_{F_a}\}$ such that $rk_M(F_1 \cup \cdots \cup F_a) \leq a+k-2$.*[2] *This matroid is called the k-th* Dilworth truncation *of M.*[3]

[2]The idea behind this is that, if the span of $F_1, \ldots, F_a$ has dimension less than $a + k - 1$, then, upon intersection with V (which has codimension $k-1$), their span will have dimension less than a.

[3]The matroid $D_k(M)$ can be defined combinatorially by specifying its circuits in the same way, even if M is not representable. In fact, when M is representable, the most subtle aspect of our definition of $D_k(M)$ is the construction of a sufficiently generic subspace V, and hence of a geometric realization of $D_k(M)$. This construction was proposed by Mason [29] and proved correct by Brylawski [7]. They also showed that, if M is not realizable, then $D_k(M)$ is not realizable either.

This is precisely the setup that we need. Let $L = \{1, \dots, m\}$ be the coordinate axes of $\mathbb{C}^m$, labelled so that coordinate hyperplane J_i is normal to axis i. These m lines are a realization of the free matroid M_m on m elements.

Now consider the $(m-n+1)$-th Dilworth truncation $D_{m-n+1}(M_m)$ of M_m, obtained by intersecting our configuration with an n-dimensional subspace V of $\mathbb{C}^m$, which is generic enough for Theorem 2.3 to apply. For each $(m-n+1)$-subset T of $L = \{1, \dots, m\}$, we get an element of the matroid of the form

$$v_T = (\text{span } T) \cap V = \Big(\bigcap_{i \notin T} J_i\Big) \cap V = \bigcap_{i \notin T} (J_i \cap V) = \bigcap_{i \notin T} H_i = H_{[m]-T},$$

where, as before, $H_i = J_i \cap V$ is a hyperplane in V. Since $\big|[m] - T\big| = n - 1$, this v_T is precisely one of the lines in the arrangement $L_{n,m}$ of one-dimensional intersections of $\{H_1, \dots, H_m\}$. In Theorem 2.3, we have a combinatorial description for the matroid $D_{m-n+1}(M_m)$ of the v_Ts. It remains to check that this matches our description of $\mathcal{H}_{n,m}$.

This verification is straightforward. In $D_{m-n+1}(M_m)$, the collection $\{v_{T_1}, \dots, v_{T_a}\}$ is a circuit if it is a minimal set such that the following equivalent conditions hold:

$$\begin{aligned} \mathrm{rk}_{M_m}(T_1 \cup \cdots \cup T_a) &\leq a + (m-n+1) - 2, \\ |T_1 \cup \cdots \cup T_a| &\leq m - (n-a+1), \\ |([m]-T_1) \cap \cdots \cap ([m]-T_a)| &\geq n - a + 1. \end{aligned}$$

This is equivalent to $\{[m]-T_1, \dots, [m]-T_a\}$ being a circuit of the matroid $\mathcal{H}_{n,m}$, which is precisely what we wanted to show. This completes the proof of Theorem 2.2. □

Corollary 2.4. *The matroid $\mathcal{H}_{n,m}$ is isomorphic to the $(m-n+1)$-th Dilworth truncation of the free matroid M_m.*

Proof. This is an immediate consequence of our proof of Theorem 2.2. □

Comment. Given a hyperplane arrangement $\mathcal{A}$, Manin and Schechtman [27] and Bayer and Brandt [5] studied the space $U(\mathcal{A})$ of arrangements of hyperplanes which are in the most general position possible, while staying parallel to the hyperplanes of $\mathcal{A}$. They showed that this space is itself the complement of a central hyperplane arrangement $\mathcal{B}(\mathcal{A})$, called the *discriminantal arrangement* of $\mathcal{A}$.

Their construction is closely related to ours, as observed by Falk [14] and Bayer and Brandt [5]. Let $\mathcal{H}$ and $\mathcal{H}^*$ be dual hyperplane arrangements in the matroid sense. Then the arrangement of lines determined by $\mathcal{H}$ is linearly isomorphic to the arrangement of lines normal to the discriminantal arrangement of $\mathcal{H}^*$. In particular, Theorem 2.2 follows from this circle of ideas; see [2, 9, 14, 27].

3 From lines in a flag arrangement to lattice points in a simplex.

Having understood the matroid of lines in a generic hyperplane arrangement, we proceed to study the case of complete flags. In the following three sections, we will describe the matroid of lines of a generic arrangement of d complete flags in $\mathbb{C}^n$. We start, in this section, with a short discussion of the combinatorial setup that we will use to encode these geometric objects. We then propose, in Section 4, a combinatorial definition of the matroid $\mathcal{T}_{n,d}$. Finally, we will show in Section 5 that this is, indeed, the matroid we are looking for.

Let $E^1_\bullet, \ldots, E^d_\bullet$ be d generically chosen complete flags in $\mathbb{C}^n$. Write

$$E^k_\bullet = \{\{0\} = E^k_0 \subset E^k_1 \subset \cdots \subset E^k_n = \mathbb{C}^n\},$$

where E^k_i is a vector space of dimension i.

These d flags determine a line arrangement $\mathbf{E}_{n,d}$ in $\mathbb{C}^n$ as follows. Look at all the possible intersections of the subspaces under consideration; they are of the form $E_{a_1,\ldots,a_d} = E^1_{a_1} \cap E^2_{a_2} \cap \cdots \cap E^d_{a_d}$. We are interested in the one-dimensional intersections. Since the $E^k_\bullet$s were chosen generically, $E_{a_1,\ldots,a_d}$ has codimension $(n-a_1)+\ldots+(n-a_d)$ (or n if this sum exceeds n). Therefore, the one-dimensional intersections are the lines $E_{a_1,\ldots,a_d}$ for $a_1+\cdots+a_d = (d-1)n+1$. There are $\binom{n+d-2}{d-1}$ such lines, corresponding to the ways of writing $n-1$ as a sum of d nonnegative integers $n-a_1,\ldots,n-a_d$.

Let $T_{n,d}$ be the set of lattice points in the following $(d-1)$-dimensional simplex in $\mathbb{R}^d$:

$$\{\,(x_1,\ldots,x_d) \in \mathbb{R}^d \mid x_1+\cdots+x_d = n-1 \text{ and } x_i \geq 0 \text{ for all } i\}.$$

The d vertices of this simplex are $(n-1,0,0,\ldots,0), (0,n-1,0,\ldots,0), \ldots, (0,0,\ldots,n-1)$. For example, $T_{n,3}$ is simply a triangular array of dots *of size* n; that is, with n dots on each side. We will call $T_{n,d}$ the $(d-1)$-*simplex of size* n. Each edge contains n dots.

It will be convenient to identify the line $E_{a_1,\dots,a_d}$ (where $a_1+\cdots+a_d = (d-1)n+1$ and $1 \le a_i \le n$) with the vector $(n-a_1,\dots,n-a_d)$ of codimensions. This clearly gives us a one-to-one correspondence between the set $T_{n,d}$ and the lines in our line arrangement $\mathbf{E}_{n,d}$.

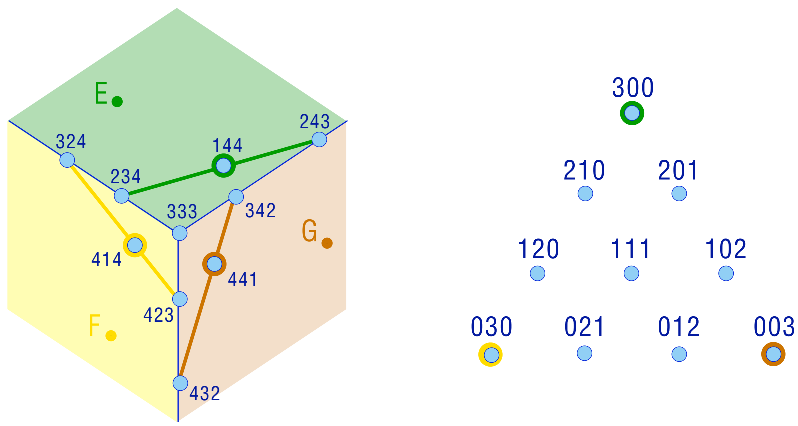

Figure 2: The lines determined by three flags in $\mathbb{C}^4$, and the array $T_{4,3}$.

We illustrate this correspondence for $d=3$ and $n=4$ in Figure 2. This picture is easier to visualize in real projective 3-space. Now each one of the flags $E_\bullet, F_\bullet$, and $G_\bullet$ is represented by a point in a line in a plane. The lines in our line arrangement are now the 10 intersection points we see in the picture.

We are interested in the dependence relations among the lines in the line arrangement $\mathbf{E}_{n,d}$. As in the case of hyperplane arrangements, there are several *combinatorial relations* which arise as follows. Consider a k-dimensional subspace $E_{b_1,\dots,b_d}$ with $b_1+\cdots+b_d=(d-1)n+k$. Every line of the form $E_{a_1,\dots,a_d}$ with $a_i \le b_i$ is in this subspace, so no $k+1$ of them can be independent. The corresponding points $(n-a_1,\dots,n-a_d)$ are the lattice points inside a parallel translate of $T_{k,d}$, the simplex of size k, in $T_{n,d}$. In other words, in a set of independent lines of our arrangement, we cannot have more than k lines whose corresponding dots are in a simplex of size k in $T_{n,d}$.

For example, no four of the lines $E_{144}, E_{234}, E_{243}, E_{324}, E_{333}$, and E_{342} are independent, because they are in the 3-dimensional hyperplane E_{344}. The dots corresponding to these six lines form the upper $T_{3,3}$ found in our $T_{4,3}$ drawn in Figure 2.

In principle, there could be other hidden dependence relations among the lines in $\mathbf{E}_{n,d}$. The goal of the next two sections is to show that this is not the case. In fact, these combinatorial relations are the only dependence relations of the line arrangement associated to d generically chosen flags in $\mathbb{C}^n$.

We will proceed as in the case of hyperplane arrangements. We will start by showing, in Section 4, that the combinatorial relations do give rise to a matroid $\mathcal{T}_{n,d}$. In Section 5, we will then show that this is, indeed, the matroid we are looking for.

4 A matroid on the lattice points in a regular simplex.

The combinatorial dependence relations defined in Section 3 do in fact determine a matroid. This will follow as a consequence of Theorem 5.1. As we did in Section 2 for hyperplanes, we will now give an alternative combinatorial proof of this statement, which is helpful in understanding the structure of the matroid we are interested in.

Theorem 4.1. *Let $\mathcal{I}_{n,d}$ be the collection of subsets I of $T_{n,d}$ such that every parallel translate of $T_{k,d}$ contains at most k points of I, for every $k \leq n$.*

Then $\mathcal{I}_{n,d}$ is the collection of independent sets of a matroid $\mathcal{T}_{n,d}$ on the ground set $T_{n,d}$.

We will call a parallel translate of $T_{k,d}$ a *simplex of size k*. As an example, $T_{n,3}$ is a triangular array of dots *of size n*. The collection $\mathcal{I}_{n,3}$ consists of those subsets I of the array $T_{n,3}$ such that no triangle of size k contains more than k points of I. Figure 3 shows the array $T_{4,3}$, and a set in $\mathcal{I}_{4,3}$.

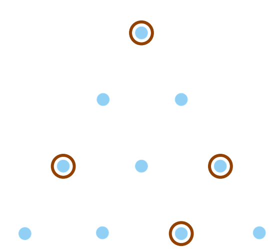

Figure 3: The array $T_{4,3}$ and a set in $\mathcal{I}_{4,3}$.

Proof of Theorem 4.1. We need to verify the three axioms for the collection of independent sets of a matroid:

(I1) The empty set is in $\mathcal{I}_{n,d}$.

(I2) If I is in $\mathcal{I}_{n,d}$ and $I' \subseteq I$, then I' is also in $\mathcal{I}_{n,d}$.
(I3) If I and J are in $\mathcal{I}_{n,d}$ and $|I| < |J|$, then there is an element e in $J - I$ such that $I \cup e$ is in $\mathcal{I}_{n,d}$.

The first two axioms are satisfied trivially; let us focus on the third one. Proceed by contradiction. Let $J - I = \{e_1, \ldots, e_m\}$. We know that every simplex of size a contains at most a points of I. When we try to add e_h to I while preserving this condition, only one thing can stop us: a simplex T_h of size t_h which already contains t_h points of I, and also contains e_h.

Say that a simplex of size t is *I-saturated* if it contains exactly t points of I. We have found I-saturated simplices $T_1, \ldots, T_m$ which contain $e_1, \ldots, e_m$, respectively.

Now we use the following lemma, which we will prove in a moment.

Lemma 4.2. *Let T and T' be two I-saturated simplices, and let $T \vee T'$ be the smallest simplex containing both of them. Suppose that T and T' are either overlapping or neighboring; that is, either*

1. *$T \cap T' \neq \emptyset$, or*
2. *$T \cap T' = \emptyset$ and $size(T \vee T') = size(T) + size(T')$.*

Then the simplices $T \cap T'$ and $T \vee T'$ are also I-saturated.

If two of our I-saturated simplices T_g and T_h are different and have a non-empty intersection, we can replace them both by $T_g \vee T_h$. By Lemma 4.2, this is also an I-saturated simplex, and it still contains e_g and e_h. We can continue in this way, until we obtain I-saturated simplices $T'_1, \ldots, T'_m$ containing $e_1, \ldots, e_m$ *which are pairwise disjoint* (though possibly repeated).

Let $U_1, \ldots, U_l$ be this collection of I-saturated simplices, now listed without repetitions. Let U_r have size s_r, and say it contains i_r elements of $I - J$, j_r elements of $J - I$, and h_r elements of $I \cap J$.

We know that U_r is I-saturated, so $s_r = i_r + h_r$. We also know that J is in $\mathcal{I}_{n,d}$, so $s_r \geq j_r + h_r$. Therefore, $i_r \geq j_r$ for each r.

Now, the U_rs are pairwise disjoint, so $\sum i_r \leq |I - J|$ and $\sum j_r \leq |J - I|$. But in fact, we know that every element of $J - I$ is in some U_r, so we actually have the equality $\sum j_r = |J - I|$. Therefore we have

$$|J - I| = \sum j_r \leq \sum i_r \leq |I - J|.$$

This contradicts our assumption that $|I| < |J|$, and Theorem 4.1 follows. $\square$

Proof of Lemma 4.2. First we show that $\text{size}(T \cap T') + \text{size}(T \vee T') = \text{size}(T) + \text{size}(T')$. This is trivial in the second case of the lemma, so we assume that $T \cap T' \neq \emptyset$.

Each simplex is a parallel translate of some $T_{k,d}$; its vertices are given by $(a_1 + k - 1, a_2, \ldots, a_d), \ldots, (a_1, a_2, \ldots, a_d + k - 1)$ for some $a_1, \ldots, a_d$ such that $\sum a_i = n - k$. We denote this simplex by $T_{a_1,\ldots,a_d}$; its size is $k = n - \sum a_i$ provided $\sum a_i \leq n$. It consists of the points $(x_1, \ldots, x_d)$ with $x_i \geq a_i$ for each i, and $\sum x_i = n - 1$. Therefore, $T_{a_1,\ldots,a_d} \subseteq T_{A_1,\ldots,A_d}$ if and only if $a_i \geq A_i$ for each i.

It follows that if $T = T_{a_1,\ldots,a_d}$ and $T' = T_{a'_1,\ldots,a'_d}$ are two overlapping simplices, then we have:

$$\begin{aligned} T \cap T' &= T_{\max(a_1,a'_1),\ldots,\max(a_d,a'_d)} \\ T \vee T' &= T_{\min(a_1,a'_1),\ldots,\min(a_d,a'_d)}. \end{aligned}$$

So $\text{size}(T \cap T') + \text{size}(T \vee T') = (n - \sum \max(a_i, a'_i)) + (n - \sum \min(a_i, a'_i))$ and $\text{size}(T) + \text{size}(T') = (n - \sum a_i) + (n - \sum a'_i)$. These are equal since $\max(a, a') + \min(a, a') = a + a'$ for any $a, a' \in \mathbb{R}$.

We know that T and T' are I-saturated, hence they contain $\text{size}(T)$ and $\text{size}(T')$ points of I, respectively. Assume that $T \cap T'$ and $T \vee T'$ contain x and y points of I. Then since we shown that $\text{size}(T \cap T') + \text{size}(T \vee T') = \text{size}(T) + \text{size}(T')$, we have that $x + y \geq \text{size}(T) + \text{size}(T') = \text{size}(T \cap T') + \text{size}(T \vee T')$. But I is in $\mathcal{I}_{n,d}$, so $x \leq \text{size}(T \cap T')$ and $y \leq \text{size}(T \vee T')$. This can only happen if equality holds, and $T \cap T'$ and $T \vee T'$ are I-saturated. $\square$

5 This is the right matroid.

We now show that the matroid $\mathcal{T}_{n,d}$ of Section 4 is, indeed, the matroid that arises from intersecting d flags in $\mathbb{C}^n$ which are generic enough.

Theorem 5.1. *If d complete flags $E^1_\bullet, \ldots, E^d_\bullet$ in $\mathbb{C}^n$ are generic enough, then the matroid of the $\binom{n+d-2}{d-1}$ lines $E_{a_1,\ldots,a_d}$ is isomorphic to $\mathcal{T}_{n,d}$.*

Proof. As mentioned in Section 3, the one-dimensional intersections of the $E^i_\bullet$s satisfy the following combinatorial relations: each k dimensional subspace $E_{b_1,\ldots,b_d}$ with $b_1 + \cdots + b_d = (d-1)n + k$, contains the lines $E_{a_1,\ldots,a_d}$ with $a_i \leq b_i$; therefore, it is impossible for $k+1$ of these lines to be independent. The subspace $E_{b_1,\ldots,b_d}$ corresponds to the simplex of dots which is labelled $T_{n-b_1,\ldots,n-b_d}$, and has size $n - \sum(n - b_i) = k$. The lines $E_{a_1,\ldots,a_d}$ with $a_i \leq b_i$ correspond precisely with the dots in this copy of $T_{k,d}$. So these "combinatorial relations" are precisely the dependence relations of $\mathcal{T}_{n,d}$.

Now we need to show that, if the flags are generic enough, these are the only linear relations among these lines. It is enough to construct one set of flags which satisfies no other relations.

Consider a set $\mathcal{H}$ of $d(n-1)$ hyperplanes H^i_j in $\mathbb{C}^n$ (for $1 \leq i \leq d$ and $1 \leq j \leq n-1$) which are generic in the sense of Theorem 2.2, so the only dependence relations among their one-dimensional intersections are the combinatorial ones. Now, for $i = 1, \ldots, d$, define the flag $E^i_\bullet$ by:

$$\begin{aligned} E^i_{n-1} &= H^i_{n-1} \\ E^i_{n-2} &= H^i_{n-1} \cap H^i_{n-2} \\ &\vdots \\ E^i_1 &= H^i_{n-1} \cap H^i_{n-2} \cap \cdots \cap H^i_1, \end{aligned}$$

We will show that these d flags are generic enough; in other words, the matroid of their one-dimensional intersections is $\mathcal{T}_{n,d}$.

Let us assume that a set S of one-dimensional intersections of the $E^i_\bullet$s is dependent. Since each line in S is a one-dimensional intersection of the hyperplanes H^i_j, we can apply Theorem 2.2. It tells us that for some t we can find $t+1$ lines in S and a set T of $n-t$ hyperplanes H^i_j which contain all of them.

Our $t+1$ lines are of the form

$$\begin{aligned} E_{a_1,\ldots,a_d} &= E^1_{a_1} \cap \cdots \cap E^d_{a_d} \\ &= (H^1_{n-1} \cap \cdots \cap H^1_{a_1}) \cap \cdots \cap (H^d_{n-1} \cap \cdots \cap H^d_{a_d}). \end{aligned}$$

Therefore, if a hyperplane H^i_j contains them, so does H^i_k for any $k > j$. Let us add all such hyperplanes to our set T, to obtain the set

$$U = \{H^1_{n-1}, \ldots, H^1_{b_1}, \ldots, H^d_{n-1}, \ldots, H^d_{b_d}\},$$

where b_i is the smallest j for which H^i_j is in T. The set U contains $\sum(n-b_i)$ hyperplanes, so $\sum(n-b_i) \geq n-t$.

Each one of our $t+1$ lines is contained in each of the hyperplanes in U, and therefore in their intersection

$$\bigcap_{H^i_j \in U} H^i_j = E_{b_1,\ldots,b_d},$$

which has dimension $n - \sum(n-b_i) \leq t$.

So, actually, the dependence of the set S is a consequence of one of the combinatorial dependence relations present in $\mathcal{T}_{n,d}$. The desired result follows. $\square$

With Theorem 5.1 in mind, we will say that the complete flags $E^1_\bullet, \ldots, E^d_\bullet$ in $\mathbb{C}^n$ are *matroid-generic* if the matroid of the $\binom{n+d-2}{d-1}$ lines $E_{a_1,\ldots,a_d}$ is isomorphic to $\mathcal{T}_{n,d}$.

We conclude this section by showing that the one-dimensional intersections of matroid-generic flag arrangements are completely characterized by their combinatorial properties.

Proposition 5.2. *If a line arrangement $\mathcal{L}$ in $\mathbb{C}^n$ has matroid $\mathcal{T}_{n,d}$, then it can be realized as the arrangement of one-dimensional intersections of d complete flags in $\mathbb{C}^n$.*

Proof. To make the notation clearer, let us give the proof for $d = 3$, which generalizes trivially to larger values of d. Denote the lines in $\mathcal{L}$ by L_{rst} for $r + s + t = 2n + 1$. Consider the three flags $E_\bullet, F_\bullet$ and $G_\bullet$ given by

$$\begin{aligned} E_i &= \operatorname{span}\{L_{rst} \mid r \leq i\} \\ F_i &= \operatorname{span}\{L_{rst} \mid s \leq i\} \\ G_i &= \operatorname{span}\{L_{rst} \mid t \leq i\} \end{aligned}$$

for $0 \leq i \leq n$. Compare this with Figure 2 in the case $n = 4$. The subspace E_i, for example, is the span of the lines corresponding to the first i rows of the triangle.

Since $\mathcal{L}$ is a representation of the matroid $\mathcal{T}_{n,3}$, the dimensions of E_i, F_i, and G_i are equal to i, which is the rank of the corresponding sets (copies of $T_{i,3}$) in $\mathcal{T}_{n,3}$.

We now claim that the line arrangement corresponding to $E_\bullet, F_\bullet$ and $G_\bullet$ is precisely $\mathcal{L}$. This amounts to showing that $E_i \cap F_j \cap G_k = L_{ijk}$ for $i + j + k = 2n + 1$. We know that L_{ijk} is in E_i, F_j, and G_k by definition, so we simply need to show that $\dim(E_i \cap F_j \cap G_k) = 1$.

Assume $\dim(E_i \cap F_j \cap G_k) \geq 2$. Consider the sequence of subspaces:

$$\begin{array}{lclclclcl} E_i \cap F_j \cap G_k & \subseteq & E_{i+1} \cap F_j \cap G_k & \subseteq & \cdots & \subseteq & E_n \cap F_j \cap G_k & \subseteq \\ & \subseteq & E_n \cap F_{j+1} \cap G_k & \subseteq & \cdots & \subseteq & E_n \cap F_n \cap G_k & \subseteq \\ & \subseteq & E_n \cap F_n \cap G_{k+1} & \subseteq & \cdots & \subseteq & E_n \cap F_n \cap G_n. & \end{array}$$

There are $1 + (n - i) + (n - j) + (n - k) = n$ subspaces on this list; the first one has dimension at least 2, and the last one has dimension n. By the pigeonhole principle, two consecutive subspaces on this list must have the same dimension. Since one is contained in the other, these two subspaces must actually be equal. So assume that $E_{a-1} \cap F_b \cap G_c = E_a \cap F_b \cap G_c$; a similar argument will work in the other cases.

Now, we have $a + b + c > i + j + k = 2n + 1$, so we can find positive integers $\beta \leq b$ and $\gamma \leq c$ such that $a + \beta + \gamma = 2n + 1$. Then $L_{a\beta\gamma}$ is a line which, by definition, is in E_a, F_b and G_c. It follows that

$$L_{a\beta\gamma} \in E_a \cap F_b \cap G_c = E_{a-1} \cap F_b \cap G_c \subseteq E_{a-1}.$$

This implies that $L_{a\beta\gamma}$ is dependent on $\{L_{rst} \mid r \leq a-1\}$, which is impossible since $\mathcal{L}$ represents the matroid $\mathcal{T}_{n,3}$. We have reached a contradiction, which implies that $\dim(E_i \cap F_j \cap G_k) = 1$ and therefore $E_i \cap F_j \cap G_k = L_{ijk}$.

It follows that $\mathcal{L}$ is the line arrangement determined by flags $E_\bullet, F_\bullet$ and $G_\bullet$, as we wished to show. $\square$

6 Rhombus tilings of holey triangles and the matroid $\mathcal{T}_{n,3}$.

Let us change the subject for a moment.

Figure 4: $T(4)$ and the three rhombus tiles.

Let $T(n)$ be an equilateral triangle with side length n. Suppose we wanted to tile $T(n)$ using unit rhombi with angles equal to $60°$ and $120°$. It is easy to see that this task is impossible, for the following reason. Cut $T(n)$ into n^2 unit equilateral triangles, as illustrated in Figure 4; $n(n+1)/2$ of these triangles point upward, and $n(n-1)/2$ of them point downward. Since a rhombus always covers one upward and one downward triangle, we cannot use them to tile $T(n)$.

Suppose then that we make n holes in the triangle $T(n)$ by cutting out n of the upward triangles. Now we have an equal number of upward and downward triangles, and it may or may not be possible to tile the remaining shape with rhombi. Figure 5 shows a tiling of one such *holey triangle*.

The main question we address in this section is the following:

Question 6.1. *Given n holes in $T(n)$, is there a simple criterion to determine whether there exists a rhombus tiling of the holey triangle that remains?*

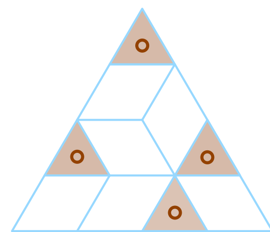

Figure 5: A tiling of a holey $T(4)$.

A rhombus tiling is equivalent to a perfect matching between the upward triangles and the downward triangles. Hall's theorem then gives us an answer to Question 6.1: It is necessary and sufficient that any k downward triangles have a total of at least k upward triangles to match to.

However, the geometry of $T(n)$ allows us to give a simpler criterion. Furthermore, this criterion reveals an unexpected connection between these rhombus tilings and the line arrangement determined by 3 generically chosen flags in $\mathbb{C}^n$. Notice that the upward triangles in $T(n)$ can be identified with the dots of $T_{n,3}$.

Theorem 6.2. *Let S be a set of n holes in $T(n)$. The triangle $T(n)$ with holes at S can be tiled with rhombi if and only if the locations of the holes constitute a basis for the matroid $\mathcal{T}_{n,3}$;* i.e., *if and only if every $T(k)$ in $T(n)$ contains at most k holes of S, for all $k \leq n$.*

Proof. First suppose that we have a tiling of the holey triangle, and consider any triangle $T(k)$ in $T(n)$. Consider all the tiles which contain one or two triangles of that $T(k)$, and let R be the holey region that these tiles cover. Since all the boundary triangles of $T(k)$ face up, the region R is just $T(k)$ with some downward triangles glued to its boundary.

If $T(k)$ had more than k holes, it would have fewer than $k(k-1)/2$ upward triangles, and so would R. However, R has at least the $k(k-1)/2$ downward triangles of $T(k)$. That makes it impossible to tile the region R, which contradicts its definition. This proves the forward direction.

Now let S be a set of n holes in $T(n)$ such that every $T(k)$ contains at most k holes. Equivalently, think of S as a basis of the matroid $\mathcal{T}_{n,3}$. We construct a tiling of the resulting holey triangle by induction on n. The case $n = 1$ is trivial, so assume $n \geq 2$.

Within that induction, we induct on the number of holes of S in the bottom row of $T(n)$. Since the $T(n-1)$ of the top $n-1$ rows contains at most $n-1$ holes, there is at least one hole in the bottom row.

If there is exactly one hole in the bottom row, then the tiling of the bottom row is forced upon us, and the top $T(n-1)$ can be tiled by induction. Now assume that there are at least two holes in the bottom row; call the two leftmost holes x and y in that order. Consider the upward triangles in the second to last row which are between x and y; label them $a_1, \ldots, a_t$. This is illustrated in an example in the top left panel of Figure 6. Here a_1, a_2, a_3 and a_4 are shaded lightly, and a_1 is also a hole.

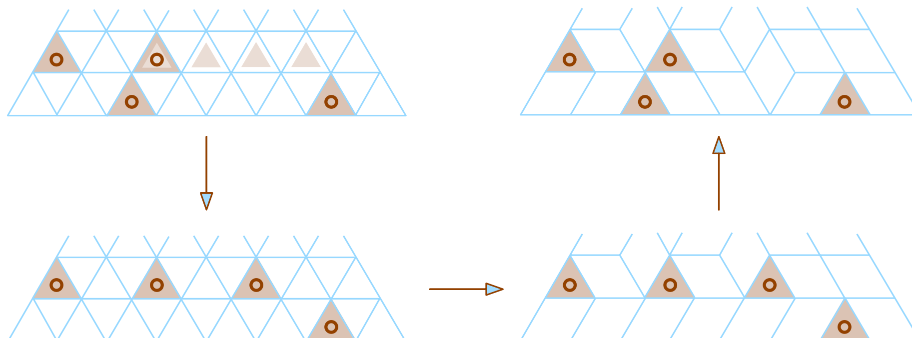

Figure 6: Sliding the hole from a_i to x.

We claim that we can exchange the hole x for one of the holes a_i, so that the set of holes $(S-x)\cup a_i$ is also a basis of $\mathcal{T}_{n,3}$. Notice that this a_i cannot be in S. Assume that no such a_i exists. Then each a_i must be in a triangle T_i which is $(S-x)$-saturated.[4] If a_i is in S, then $T_i = a_i$. The triangle y is also trivially $(S-x)$-saturated. We can then use Lemma 4.2 successively to obtain an $(S-x)$-saturated triangle containing $a_1, \ldots, a_t$, and y. But that triangle will also contain x, so it will contain more holes of S than it is allowed.

So let a_i be such that $S - x \cup a_i$ is a basis of $\mathcal{T}_{n,3}$. For instance, in the first step of Figure 6, x is exchanged for a_3. Notice that $S - x \cup a_i$ contains fewer holes in the bottom row than S does. By the induction hypothesis, we can tile the $T(n)$ with holes at $S - x \cup a_i$, as shown in the second step of Figure 6. The bottom row of this tiling is frozen from left to right until it reaches y. Therefore, we can slide the hole from a_i back to x in the obvious way, by reversing the tiles in the bottom row between x and a_i. This is illustrated in the last step of Figure 6. We are left with a tiling with holes at S, as desired. $\square$

[4] As in Section 4, if A is a set of holes, we say that an upward triangle of size k is *A-saturated* if it contains k holes of A.

Theorem 6.2 allows us to say more about the structure of the matroid $\mathcal{T}_{n,3}$. We first remind the reader of the definition of two important families of matroids, called *transversal* and *cotransversal* matroids. For more information, we refer the reader to [1, 32].

Let S be a finite set, and let $A_1, \ldots, A_r$ be subsets of S. A *transversal* of $(A_1, \ldots, A_r)$, also known as a *system of distinct representatives*, is a subset $\{e_1, \ldots, e_r\}$ of S such that e_i is in A_i for each i, and the e_is are distinct. The transversals of $(A_1, \ldots, A_r)$ are the bases of a matroid on S. Such a matroid is called a *transversal matroid*, and $(A_1, \ldots, A_r)$ is called a *presentation* of the matroid.

Let G be a directed graph with vertex set V, and let $A = \{v_1, \ldots, v_r\}$ be a subset of V. We say that an r-subset B of V *can be linked to* A if there exist r vertex-disjoint directed paths whose initial vertex is in B and whose final vertex is in A. We will call these r paths a *routing* from B to A. The collection of r-subsets which can be linked to A are the bases of a matroid denoted $L(G, A)$. Such a matroid is called a *cotransversal* matroid or a *strict gammoid*. It is a nontrivial fact that these matroids are precisely the duals of the transversal matroids [1, 32].

Theorem 6.3. *The matroid $\mathcal{T}_{n,3}$ is cotransversal.*

First proof. We prove that $\mathcal{T}^*_{n,3}$ is transversal. We can think of the ground set of $\mathcal{T}_{n,3}$ as the set of upward triangles in $T(n)$. By Theorem 6.2, a basis of $\mathcal{T}_{n,3}$ is a set of n holes for which the resulting holey triangle can be tiled; its complement is the set of $\binom{n}{2}$ upward triangles which share a tile with one of the $\binom{n}{2}$ downward triangles.

Number the downward triangles $1, 2, \ldots, N = \binom{n}{2}$. Then a tiling of the complement of a basis of $\mathcal{T}_{n,3}$ is nothing but a transversal of $(A_1, \ldots, A_N)$, where A_i is the set of three upward triangles which are adjacent to downward triangle i. This completes the proof. $\square$

Second proof. We prove that $\mathcal{T}_{n,3}$ is cotransversal. Let G_n be the directed graph whose set of vertices is the triangular array $T_{n,3}$, where each dot not on the bottom row is connected to the two dots directly below it. Label the dots on the bottom row $1, 2, \ldots, n$. Figure 7 shows G_4; all the edges of the graph point down.

We now recall a trick, commonly used in the tilings literature and attributed to Dana Randall, to convert tilings into routings; see for example [26]. In our particular situation, it allows us to view rhombus tilings of the holey triangle $T(n)$ as routings in G_n. The trick works as follows: A copy of

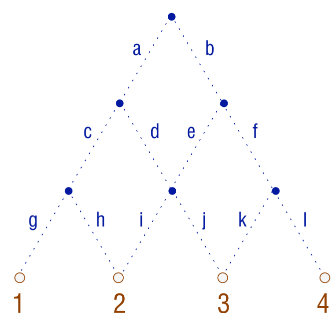

Figure 7: The graph G_4.

the graph G_n can be drawn whose vertices are the midpoints of the possible horizontal edges of a tiling. Given a tiling of a holey $T(n)$, join two vertices of G_n if they are on opposite edges of the same tile; this gives the desired routing of G_n. This correspondence is best understood in an example; see Figure 8.

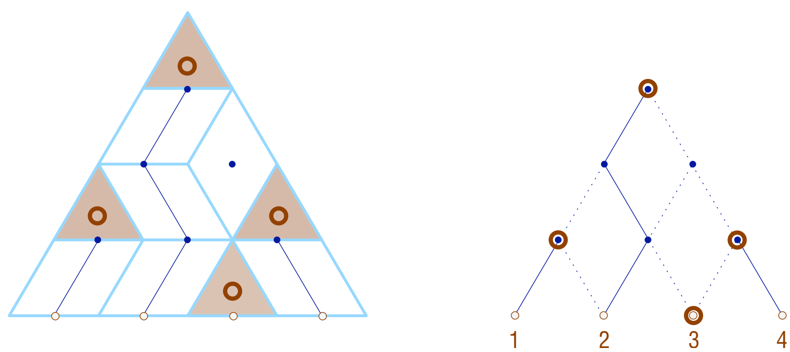

Figure 8: A tiling of a holey $T(4)$ and the corresponding routing of G_4.

Given such a routing, one can easily recover the tiling that gave rise to it: simply place one rhombus over each edge in the routing, and one vertical rhombus over each isolated vertex. It is easy to check that this is a bijection between the rhombus tilings of the holey triangles of size n, and the routings in the graph G_n which start anywhere and end at vertices $1, 2, \ldots, n$.

Notice also that, in this bijection, the holes of the holey triangle correspond to the starting points of the n paths in the routing. From Theorem 6.2, it follows that $\mathcal{T}_{n,3}$ is the cotransversal matroid $L(G_n, [n])$. $\square$

Theorem 6.4. *Assign sufficiently generic weights to the edges of G_n.*[5] *For each dot D in the triangular array $T_{n,3}$ and each $1 \le i \le n$, let $v_{D,i}$ be the sum of the weights of all paths*[6] *from dot D to dot i on the bottom row.*

Then the path vectors $v_D = (v_{D,1}, \dots, v_{D,n})$ *are a geometric representation of the matroid $T_{n,3}$.*

For example, the top dot of $T_{4,3}$ in Figure 7 would be assigned the *path vector* $(acg, ach + adi + bei, adj + bej + bfk, bfl)$. Similarly, focusing our attention on the top three rows, the representation we obtain for the matroid $\mathcal{T}_{3,3}$ is given by the columns of the following matrix:

$$\begin{pmatrix} 1 & 0 & 0 & c & 0 & ac \\ 0 & 1 & 0 & d & e & ad+be \\ 0 & 0 & 1 & 0 & f & bf \end{pmatrix}$$

Proof of Theorem 6.4. By the Lindström-Gessel-Viennot lemma [17, 20, 25, 30], the determinant of the matrix with columns $v_{D_1}, \dots, v_{D_n}$ is equal to the signed sum of the routings from $\{D_1, \dots, D_n\}$ to $\{1, \dots, n\}$. The sign of a routing is the sign of the permutation of S_n which matches the starting points and the ending points of the n paths. For sufficiently generic weights, this signed sum can only equal zero if it is empty.

Therefore, $v_{D_1}, \dots, v_{D_n}$ are independent if and only if there exists a routing from $\{D_1, \dots, D_n\}$ to $\{1, \dots, n\}$. This is equivalent to $\{D_1, \dots, D_n\}$ being a basis of $L(G_n, [n])$. $\square$

It is worth pointing out that Lindström's original motivation for the discovery of the Lindström-Gessel-Viennot lemma was to explain Mason's construction of a geometric representation of an arbitrary cotransversal matroid [25, 29]. Theorem 6.4 and its proof are special cases of their more general argument; we have included them for completeness.

The very simple and explicit representation of $\mathcal{T}_{n,3}$ of Theorem 6.4 will be shown in Section 9 to have an unexpected consequence in the Schubert calculus: it provides us with a reasonably efficient method for computing Schubert structure constants in the flag manifold.

[5] We will see that it is enough to choose weights in a certain Zariski open set.

[6] The weight of a path is defined to be the product of the weights of its edges.

7 Fine mixed subdivisions of $n\Delta_{d-1}$ and triangulations of $\Delta_{n-1} \times \Delta_{d-1}$.

The surprising relationship between the geometry of three flags in $\mathbb{C}^n$ and the rhombus tilings of holey triangles is useful to us in two ways: it explains the structure of the matroid $\mathcal{T}_{n,3}$, and it clarifies the conditions for a rhombus tiling of such a region to exist. We now investigate a similar connection between the geometry of d flags in $\mathbb{C}^n$, and certain $(d-1)$-dimensional analogs of these tilings, known as *fine mixed subdivisions* of $n\Delta_{d-1}$.

The fine mixed subdivisions of $n\Delta_{d-1}$ are in one-to-one correspondence with the triangulations of the polytope $\Delta_{n-1} \times \Delta_{d-1}$. The triangulations of a product of two simplices are fundamental objects, which have been studied from many different points of view. They are of independent interest [3, 4, 16], and have been used as a building block for finding efficient triangulations of high dimensional cubes [18, 31] and disconnected flip-graphs [40, 41]. They also arise very naturally in connection with tropical geometry [11], transportation problems, and Segre embeddings [42]. In the following two sections, we provide evidence that triangulations of $\Delta_{n-1} \times \Delta_{d-1}$ are also closely connected to the geometry of d flags in $\mathbb{C}^n$, and that their study can be regarded as a study of *tropical oriented matroids*.

Instead of thinking of rhombus tilings of a holey triangle, it will be slightly more convenient to think of them as *lozenge tilings* of the triangle: these are the tilings of the triangle using unit rhombi and upward unit triangles. A good high-dimensional analogue of the lozenge tilings of the triangle $n\Delta_2$ are the *fine mixed subdivisions* of the simplex $n\Delta_{d-1}$; we briefly recall their definition.

The *Minkowski sum* of polytopes $P_1, \ldots, P_k$ in $\mathbb{R}^m$, is:

$$P = P_1 + \cdots + P_k := \{p_1 + \cdots + p_k \,|\, p_1 \in P_1, \ldots, p_k \in P_k\}.$$

We are interested in the Minkowski sum $n\Delta_{d-1}$ of n simplices. Define a *fine mixed cell* of this sum $n\Delta_{d-1}$ to be a Minkowski sum $B_1 + \cdots + B_n$, where the B_is are faces of Δ_{d-1} which lie in independent affine subspaces, and whose dimensions add up to $d-1$. A *fine mixed subdivision* of $n\Delta_{d-1}$ is a subdivision[7] of $n\Delta_{d-1}$ into fine mixed cells [38, Theorem 2.6].

Consider the case $d = 3$. If the vertices of Δ_2 are labelled A, B, and C, there are two different kinds of fine mixed cells: a unit triangle like

[7] A subdivision of a polytope P is a tiling of P with polyhedral cells whose vertices are vertices of P, such that the intersection of any two cells is a face of both of them.

$ABC+A+B+\cdots+A$, and a unit rhombus like $AB+AC+A+\cdots+C$ (which can face in three possible directions). Therefore the fine mixed subdivisions of the triangle $n\Delta_2$ are precisely its lozenge tilings. In these sums, the summands which are not points determine the shape of the fine mixed cell, while the summands which are points translate that cell inside $n\Delta_2$. This is illustrated in the right hand side of Figure 9: a lozenge tiling of $2\Delta_2$ whose tiles are $ABC+B$, $AC+AB$, and $C+ABC$.

For $d=4$, if we label the tetrahedron $ABCD$, we have four congruence classes of fine mixed cells: tetrahedra like $ABCD+A+\cdots$, triangular prisms like $ABC+AD+A+\cdots$, and two different classes of parallelepipeds: $AB+AC+AD+A+\cdots$ and $AB+BC+CD+A+\cdots$.

In the same way that we identified arrays of triangles with triangular arrays of dots in Section 6, we can identify the array of possible locations of the simplices in $n\Delta_{d-1}$ with the array of dots $T_{n,d}$ defined in Section 3. A conjectural generalization of Theorem 6.2, which we now state, would show that fine mixed subdivisions of $n\Delta_{d-1}$ are also closely connected to the matroid $\mathcal{T}_{n,d}$.

Conjecture 7.1. *The possible locations of the simplices in a fine mixed subdivision of $n\Delta_{d-1}$ are precisely the bases of the matroid $\mathcal{T}_{n,d}$.*

In the remainder of this section, we will give a completely combinatorial description of the fine mixed subdivisions of $n\Delta_{d-1}$. Then, in Section 8, we will use this description to prove Proposition 8.2, which is the forward direction of Conjecture 7.1.

We start by recalling the one-to-one correspondence between the fine mixed subdivisions of $n\Delta_{d-1}$ and the triangulations of $\Delta_{n-1}\times\Delta_{d-1}$. This equivalent point of view has the drawback of bringing us to a higher-dimensional picture. Its advantage is that it simplifies greatly the combinatorics of the tiles, which are now just simplices.

Let $v_1,\ldots,v_n$ and $w_1,\ldots,w_d$ be the vertices of Δ_{n-1} and Δ_{d-1}, so that the vertices of $\Delta_{n-1}\times\Delta_{d-1}$ are of the form $v_i\times w_j$. A triangulation T of $\Delta_{n-1}\times\Delta_{d-1}$ is given by a collection of simplices. For each simplex t in T, consider the fine mixed cell whose i-th summand is $w_a w_b\ldots w_c$, where $a,b,\ldots,c$ are the indexes j such that $v_i\times w_j$ is a vertex of t. These fine mixed cells constitute the fine mixed subdivision of $n\Delta_{d-1}$ corresponding to T. (This bijection is only a special case of the more general Cayley trick, which is discussed in detail in [38].)

For instance, Figure 9 shows a triangulation of the triangular prism $\Delta_1\times\Delta_2=12\times ABC$, and the corresponding fine mixed subdivision of $2\Delta_2$, whose three tiles are $ABC+B, AC+AB$, and $C+ABC$.

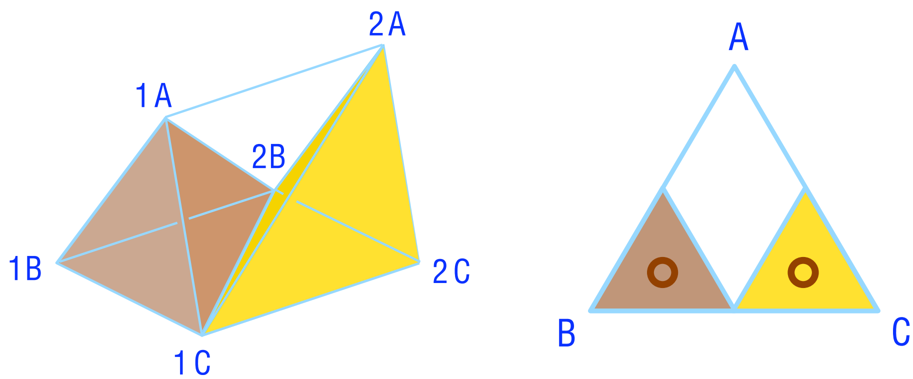

Figure 9: The Cayley trick.

Consider the complete bipartite graph $K_{n,d}$ whose vertices are $v_1, \ldots, v_n$ and $w_1, \ldots, w_d$. Each vertex of $\Delta_{n-1} \times \Delta_{d-1}$ corresponds to an edge of $K_{n,d}$. The vertices of each simplex in $\Delta_{n-1} \times \Delta_{d-1}$ determine a subgraph of $K_{n,d}$. Each triangulation of $\Delta_{n-1} \times \Delta_{d-1}$ is then encoded by a collection of subgraphs of $K_{n,d}$. Figure 10 shows the three trees that encode the triangulation of Figure 9.

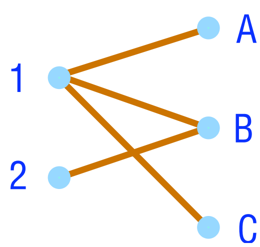

Figure 10: The trees corresponding to the triangulation of Figure 9.

Our next result is a combinatorial characterization of the triangulations of $\Delta_{n-1} \times \Delta_{d-1}$.

Proposition 7.2. *A collection of subgraphs $t_1, \ldots, t_k$ of $K_{n,d}$ encodes a triangulation of $\Delta_{n-1} \times \Delta_{d-1}$ if and only if:*

1. *Each t_i is a spanning tree.*
2. *For each t_i and each internal*[8] *edge e of t_i, there exists an edge f and a tree t_j with $t_j = (t_i - e) \cup f$.*

[8] An edge of a tree is *internal* if it is not adjacent to a leaf.

3. *There do not exist two trees t_i and t_j, and a circuit C of $K_{n,d}$ which alternates between edges of t_i and edges of t_j.*

Proof. If $e_1, \ldots, e_n, f_1, \ldots, f_d$ is a basis of $\mathbb{R}^{n+d}$, then a realization of the polytope $\Delta_{n-1} \times \Delta_{d-1}$ is given by assigning the vertex $v_i \times w_j$ coordinates $e_i + f_j$. It is then easy to see that the oriented matroid of affine dependencies of $\Delta_{n-1} \times \Delta_{d-1}$ is the same as the oriented matroid of the graph $K_{n,d}$, with edges oriented $v_i \to w_j$ for $1 \leq i \leq n, 1 \leq j \leq d$. In other words, each minimal affinely dependent set C of vertices of $\Delta_{n-1} \times \Delta_{d-1}$ corresponds to a circuit of the graph $K_{n,d}$. Furthermore, the sets C^+ and C^- of vertices which have positive and negative coefficients in the affine dependence relation of C correspond, respectively, to the edges that the circuit of $K_{n,d}$ traverses in the forward and backward direction. Therefore, a set of vertices of $\Delta_{n-1} \times \Delta_{d-1}$ forms an $(n + d - 2)$-dimensional simplex if and only if it is encoded by a spanning tree of $K_{n,d}$.

The three conditions in the statement of Proposition 7.2 simply rephrase the following result [39, Theorem 2.4.(f)]:

> Suppose we are given a polytope P, and a non-empty collection of simplices whose vertices are vertices of P. The simplices form a triangulation of P if and only if they satisfy the *pseudo-manifold property*, and *no two simplices overlap on a circuit*.

The *pseudo-manifold property* is that, for any simplex σ and any facet τ of σ, either τ is in a facet of P, or there exists another simplex σ' with $\tau \subset \sigma'$. The facets of $\Delta_{n-1} \times \Delta_{d-1}$ are of the form $F \times \Delta_{d-1}$ for a facet F of Δ_{n-1} (obtained by deleting one of the vertices of Δ_{n-1}), or $\Delta_{n-1} \times G$ for a facet G of Δ_{d-1} (obtained by deleting one of the vertices of Δ_{d-1}). Therefore, in the simplex σ corresponding to tree t, the facet of σ corresponding to $t - e$ is in a facet of $\Delta_{n-1} \times \Delta_{d-1}$ if and only if $t - e$ has an isolated vertex. So in this case, 2. is equivalent to the pseudo-manifold property.

Two simplices σ and σ' are said to *overlap on a signed circuit* $C = (C^+, C^-)$ of P if σ contains C^+ and σ' contains C^-. The circuits of the polytope $\Delta_{n-1} \times \Delta_{d-1}$ correspond precisely to the circuits of $K_{n,d}$, which are alternating in sign. Therefore this condition is equivalent to 3. □

In light of Proposition 7.2, we will call a collection of spanning trees satisfying the above properties a triangulation of $\Delta_{n-1} \times \Delta_{d-1}$.

Proposition 7.2 is implicit in work of Kapranov, Postnikov, and Zelevinsky [34, Section 12], and Babson and Billera [3]. The latter also gave a different combinatorial description of the **regular** triangulations, which we now describe.

Recall the following geometric method for obtaining subdivisions of a polytope P in $\mathbb{R}^d$. Assign a height $h(v)$ to each vertex v of P, lift the vertex v to the point $(v, h(v))$ in $\mathbb{R}^{d+1}$, and consider the lower facets of the convex hull of those new points in $\mathbb{R}^{d+1}$. The projections of those lower facets onto the hyperplane $x_{d+1} = 0$ form a subdivision of P. Such a subdivision is called *regular* or *coherent*.

A regular subdivision of the polytope $\Delta_{n-1} \times \Delta_{d-1}$ is determined by an assignment of heights to its vertices. This is equivalent to a weight vector w consisting of a weight w_{ij} for each edge ij of $K_{n,d}$. Let a *w-weighting* be an assignment (u, v) of vertex weights $u_1, \ldots, u_n$, $v_1, \ldots, v_d$ to $K_{n,d}$ such that $u_i + v_j \geq w_{ij}$ for every edge ij of $K_{n,d}$. Say edge ij is *w-tight* if the equality $u_i + v_j = w_{ij}$ holds; these edges form the *w-tight subgraph* of (u, v). A subgraph of $K_{n,d}$ is *w-tight* if it is the w-tight subgraph of *some* w-weighting.

Proposition 7.3. *[3] Let w be a height vector for $\Delta_{n-1} \times \Delta_{d-1}$ or, equivalently, a weight vector on the edges of $K_{n,d}$. The regular subdivision corresponding to w consists of the maximal w-tight subgraphs of $K_{n,d}$.*

Say a weight vector w is *generic* if no circuit of $K_{n,d}$ has alternating sum of weights equal to 0. We leave it to the reader to check, using Proposition 7.3, that generic weight vectors are precisely the ones that give rise to regular triangulations. Hence, if w is generic, the maximal w-tight subgraphs of $K_{n,d}$ are trees, and they satisfy the conditions of Proposition 7.2. It is an instructive exercise to prove this directly.

8 Subdivisions of $n\Delta_{d-1}$ and the matroid $\mathcal{T}_{n,d}$.

Having given a combinatorial characterization of the triangulations of the polytope $\Delta_{n-1} \times \Delta_{d-1}$ in Proposition 7.2, we are now in a position to prove the forward direction of Conjecture 7.1, which relates these triangulations to the matroid $\mathcal{T}_{n,d}$. The following combinatorial lemma will play an important role in our proof.

Proposition 8.1. *Let n, d, and $a_1, \ldots, a_d$ be non-negative integers such that $a_1 + \cdots + a_d \leq n - 1$. Suppose we have a coloring of the $n(n-1)$ edges of the directed complete graph K_n with d colors, such that each color defines a poset on $[n]$; in other words,* *(a) the edges $u \to v$ and $v \to u$ have different colors, and*

(b) if $u \to v$ and $v \to w$ have the same color, then $u \to w$ has that same color.

Call a vertex v outgoing *if, for every i, there exist at least a_i vertices w such that $v \to w$ has color i. Then the number of outgoing vertices is at most $n - a_1 - \cdots - a_d$.*

Proof. We have d poset structures on the set $[n]$, and this statement says that we cannot have "too many" elements which are "very large" in all the posets.

Say there are x outgoing vertices, and let v be one of them. Let x_i be the number of i-colored edges which go from v to another outgoing vertex, so $x_1 + \ldots + x_d = x - 1$.

Consider the x_1 outgoing vertices $u_1, \ldots, u_{x_1}$ such that $v \to u_j$ is blue. The blue subgraph of K_n is a poset; so among the u_js we can find a minimal one, say u_1, in the sense that $u_1 \to u_j$ is not blue for any j. Since u_1 is outgoing, there are at least a_1 vertices w of the graph such that $u_1 \to w$ is blue. This gives us a_1 vertices w, other than the u_is, such that $v \to w$ is blue. Therefore the blue outdegree of v in K_n is at least $x_1 + a_1$.

Repeating the same reasoning for the other colors, and summing over all colors, we obtain:

$$\begin{aligned} n - 1 &= \sum_{i=1}^{d} (\text{color-}i \text{ outdegree of } v) \\ &\geq \sum_{i=1}^{d} (x_i + a_i) \\ &= x - 1 + \sum_{i=1}^{d} a_i, \end{aligned}$$

which is precisely what we wanted to show. □

Notice that the bound of Proposition 8.1 is optimal. To see this, partition $[n]$ into sets $A_1, \ldots, A_d, A$ of sizes $a_1, \ldots, a_d, n - a_1 - \cdots - a_d$, respectively. For each i, let the edges from A to A_i have color i. Let the edges from A_1 to A have color d, and the edges from the other A_is to A have color 1. Pick a linear order for A, and let the edges within A have color d in the increasing order, and color 1 in the decreasing order. Pick a linear order for $A_1 \cup \cdots \cup A_d$ where the elements of A_1 are the smallest and the elements of A_d are the largest. Let the edges within $A_1 \cup \cdots \cup A_d$ have color d in the increasing order, and color 1 in the decreasing order. It is easy to check that this coloring satisfies the required conditions, and it has exactly $n - a_1 - \cdots - a_d$ outgoing vertices.

Also notice that our proof of Proposition 8.1 generalizes almost immediately to the situation where we allow edges to be colored with more than one color.

We have now laid down the necessary groundwork to prove one direction of Conjecture 7.1.

Proposition 8.2. *In any fine mixed subdivision of $n\Delta_{d-1}$,*

(a) there are exactly n tiles which are simplices, and

(b) the locations of the n simplices give a basis of the matroid $\mathcal{T}_{n,d}$.

Proof of Proposition 8.2. Let us look back at the way we defined the correspondence between a triangulation T of $\Delta_{n-1} \times \Delta_{d-1}$ and a fine mixed subdivision $f(T)$ of $n\Delta_{d-1}$. It is clear that the simplices $f(t)$ of $f(T)$ arise from those simplices t of T whose vertices are $v_i \times w_1, \ldots, v_i \times w_d$ (for some i), and one $v_j \times w_{g(j)}$ for each $j \neq i$. Furthermore, the location of $f(t)$ in $n\Delta_{d-1}$ is given by the sum of the $w_{g(j)}$s.

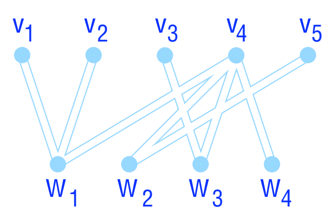

Figure 11: A spanning tree of $K_{5,4}$.

For instance the spanning tree of $K_{5,4}$ shown in Figure 11 gives rise to a simplex in a fine mixed subdivision of $5\Delta_3 = 5w_1w_2w_3w_4$ given by the Minkowski sum $w_1 + w_1 + w_3 + w_1w_2w_3w_4 + w_2$. The location of this simplex in $5\Delta_3$ corresponds to the point $(2, 1, 1, 0)$ of $T_{5,4}$, because the Minkowski sum above contains two w_1 summands, one w_2, and one w_3.

In other words, the simplices of the fine mixed subdivision of $n\Delta_{d-1}$ come from spanning trees t of $K_{n,d}$ for which one vertex v_i has degree d and the other v_js have degree 1. The coordinates of the location of $f(t)$ in $n\Delta_{d-1}$ are simply $(\deg_t w_1 - 1, \ldots, \deg_t w_d - 1)$. Call such a simplex, and the corresponding tree, *i-pure*. Figure 11 shows a 4-pure tree. Also, in the triangulation of Figures 9 and 10, there is a 1-pure tree and a 2-pure tree, which give simplices in locations $(0,1,0)$ and $(0,0,1)$ of $2\Delta_2$, respectively.

Proof of (a). We claim that in a triangulation T of $\Delta_{n-1} \times \Delta_{d-1}$ there is exactly one i-pure simplex for each i with $1 \leq i \leq n$.

First we show there is at least one i-pure simplex. If we restrict the trees of T to the "claw" subgraph $K_{\{v_i\},\{w_1,\ldots,w_d\}}$, they should encode a triangulation of the face $v_i \times (w_1 \ldots w_d)$ of $\Delta_{n-1} \times \Delta_{d-1}$. This triangulation necessarily consists of a single simplex, encoded by the claw graph. Therefore, there must be at least one spanning tree t in T containing this claw.

Now assume that we have two i-pure trees t_1 and t_2. They must differ somewhere, so assume that t_1 contains edge $v_a w_b$ and t_2 contains $v_a w_c$. Then we have a circuit $v_a w_b v_i w_c$ of $K_{n,d}$ whose edges alternate between t_1 and t_2, a contradiction.

Proof of (b). As in the proof of Lemma 4.2, let $T_{a_1,\ldots,a_d}$ be the simplex consisting of the locations $(x_1, \ldots, x_d)$ in $n\Delta_{d-1}$ such that $\sum x_i = n$ and $x_i \geq a_i$ for each i. We need to show that $T_{a_1,\ldots,a_d}$, which has a sidelength of $n - a_1 - \cdots - a_d$, contains at most $n - a_1 - \cdots - a_d$ simplices of the fine mixed subdivision.

Somewhat predictably, we will construct a coloring of the directed complete graph K_n which will allow us to invoke Proposition 8.1. This coloring will be an economical way of storing the descriptions of the n pure simplices or, equivalently, the n pure trees. Let t_i be the i-pure tree in the corresponding triangulation of $\Delta_{n-1} \times \Delta_{d-1}$. We will color the edge $i \to j$ in K_n with the color a, where w_a is the unique neighbor of vertex v_j in tree t_i. We claim that the a-colored subgraph of K_n is a poset for each color a.

First assume that $i \to j$ and $j \to i$ have the same color a. Then tree t_i contains edge $v_j w_a$ and tree t_j contains edge $v_i w_a$. Then, for any $b \neq a$, we have a circuit $v_i w_b v_j w_a$ of $K_{n,d}$ which alternates between edges of t_i and t_j, a contradiction.

Now assume that $i \to j$ and $j \to k$ have color a, but $i \to k$ has some other color b. This means that $v_j w_a$ and $v_k w_b$ are edges of t_i and $v_k w_a$ is an edge of t_j. But then the circuit $v_j w_a v_k w_b$ of $K_{n,d}$ alternates between edges of t_i and t_j, a contradiction.

We can now apply Proposition 8.1, and conclude that there are at most $n - a_1 - \ldots - a_d$ outgoing vertices in our coloring of K_n. But observe that the simplex of $n\Delta_{d-1}$ corresponding to the i-pure tree t_i is in location

$$(\deg_{t_i}(w_1) - 1, \ldots, \deg_{t_i}(w_n) - 1) = (\text{outdeg}_{K_n, \text{color } 1}(i), \ldots, \text{outdeg}_{K_n, \text{color } d}(i)).$$

Therefore the simplex of the fine mixed subdivision which corresponds to t_i is in $T_{a_1,\ldots,a_d}$ if and only if vertex i is outgoing in our coloring of K_n. The desired result follows. $\square$

For the converse of Conjecture 7.1, we would need to show that every basis of $\mathcal{T}_{n,d}$ arises from the placement of simplices in some fine mixed subdivision of $n\Delta_{d-1}$. In fact, a stronger result might hold, which we state after introducing the necessary definitions.

Recall the definition of a *regular subdivision* of a polytope given in Section 7. Similarly, a *regular mixed subdivision* of a Minkowski sum $P_1+\cdots+P_n$ in $\mathbb{R}^d$ is obtained by assigning a height $h_i(v)$ to each vertex v of P_i, and projecting the lower facets of the convex hull of the points in $\mathbb{R}^{d+1}$ of the form $(v_1, h_1(v_1)) + \cdots + (v_n, h_n(v_n))$, where v_i is a vertex of P_i.

Question 8.3. *Is it true that, for any basis B of $\mathcal{T}_{n,d}$, there is a* **regular** *fine mixed subdivision of $n\Delta_{d-1}$ whose n simplices are located at B?*

The Cayley trick provides us with a bijection between the triangulations of $\Delta_{n-1} \times \Delta_{d-1}$ and the fine mixed subdivisions of $n\Delta_{d-1}$. This correspondence also gives a bijection between regular triangulations of $\Delta_{n-1} \times \Delta_{d-1}$ and regular fine mixed subdivisions of $n\Delta_{d-1}$ [19, Theorem 3.1]. There is also a correspondence between the regular triangulations of $\Delta_{n-1}\times\Delta_{d-1}$ and the combinatorial types of arrangements of d generic tropical hyperplanes in tropical $(n-1)$-space [11, 38].

Just as the combinatorial properties of real hyperplane arrangements are captured in the theory of oriented matroids, tropical hyperplane arrangements deserve an accompanying theory of *tropical oriented matroids*. The discussion of the previous paragraph suggests that subdivisions of products of two simplices play the role of tropical oriented matroids, with regular subdivisions corresponding to realizable tropical oriented matroids. The multiple appearances of these subdivisions in the literature are presumably a good indication of the applicability of tropical oriented matroid theory. Our ability to attack Conjecture 7.1 and Question 8.3 is one way to measure our progress on this theory.

9 Applications to Schubert calculus.

In this section, we show some of the implications of our work in the Schubert calculus of the flag manifold. Throughout this section, we will assume some familiarity with the Schubert calculus, though we will recall some of the definitions and conventions that we will use; for more information, see for example [15, 28]. We will also need some of the results of Eriksson and

Linusson [12, 13] and Billey and Vakil [6] on Schubert varieties and permutation arrays.

The flag manifold $\mathcal{F}\ell_n = \mathcal{F}\ell_n(\mathbb{C})$ is a smooth projective variety which parameterizes the complete flags in $\mathbb{C}^n$. The *relative position* of any two flags $E_\bullet$ and $F_\bullet$ in $\mathcal{F}\ell_n$ is given by a permutation $w \in S_n$. Let us explain what this means.

To the permutation w, we associate the permutation matrix[9] which has a 1 in the $w(i)$th row of column $n-i+1$ for $1 \le i \le n$. Let $w[i,j]$ be the principal submatrix with lower right hand corner (i,j), and form an $n \times n$ table, called a *rank array*, whose entry (i,j) is equal to $\operatorname{rk} w[i,j]$, the number of ones in $w[i,j]$. The matrix and rank array associated to $w = 53124 \in S_5$ are shown below.

$$\begin{bmatrix} 0&0&1&0&0\\ 0&1&0&0&0\\ 0&0&0&1&0\\ 1&0&0&0&0\\ 0&0&0&0&1 \end{bmatrix} \to \begin{bmatrix} 0&0&1&1&1\\ 0&1&2&2&2\\ 0&1&2&3&3\\ 1&2&3&4&4\\ 1&2&3&4&5 \end{bmatrix}.$$

Saying that $E_\bullet$ and $F_\bullet$ are in relative position w means that the dimensions $\dim(E_i \cap F_j)$ are given precisely by the rank array of w; that is,

$$\dim(E_i \cap F_j) = \operatorname{rk} w[i,j] \qquad \text{for all} \ \ 1 \le i,j \le n.$$

Eriksson and Linusson [12, 13] introduced a higher-dimensional analog of a permutation matrix, called a *permutation array*. A permutation array is an array of dots in the cells of a d-dimensional $n \times n \times \cdots \times n$ box, satisfying some quite restrictive properties. From a permutation array P, via a simple combinatorial rule, one can construct a *rank array* of integers, also of shape $[n]^d$. We denote it $\operatorname{rk} P$. This definition is motivated by their result [13] that the relative position of d flags $E^1_\bullet, \ldots, E^d_\bullet$ in $\mathcal{F}\ell_n$ is described by a unique permutation array P, via the equations

$$\dim\bigl(E^1_{x_1} \cap \cdots \cap E^d_{x_d}\bigr) = \operatorname{rk} P[x_1, \ldots, x_d] \qquad \text{for all} \ \ 1 \le x_1, \ldots, x_d \le n.$$

This result initiated the study of *permutation array schemes*, which generalize Schubert varieties in the flag manifold $\mathcal{F}\ell_n$. These schemes are much more subtle than their counterparts; they can be empty, and are not necessarily irreducible or even equidimensional [6].

[9] Notice that this is slightly different from the usual convention, but it is useful from the point of view of permutation arrays.

The relative position of d generic flags is described by the *transversal permutation array*

$$\left\{(x_1, \ldots, x_d) \in [n]^d \;\middle|\; \sum_{i=1}^{d} x_i = (d-1)n + 1\right\}.$$

For $\sum_{i=1}^d x_i = (d-1)n+1$, the dot at position $(x_1, \ldots, x_d)$ represents a one-dimensional intersection $E^1_{x_1} \cap \cdots \cap E^d_{x_d}$. Naturally, we identify the dots in the transversal permutation array with the elements of the matroid $\mathcal{T}_{n,d}$.

Given a fixed flag $E_\bullet$, define a *Schubert cell* and *Schubert variety* to be

$$\begin{aligned}
X^\circ_w(E_\bullet) &= \{F_\bullet \,|\, E_\bullet \text{ and } F_\bullet \text{ have relative position } w\} \\
&= \{F_\bullet \mid \dim(E_i \cap F_j) = \operatorname{rk} w[i,j] \text{ for all } 1 \le i,j \le n\}, \text{ and} \\
X_w(E_\bullet) &= \{F_\bullet \mid \dim(E_i \cap F_j) \ge \operatorname{rk} w[i,j] \text{ for all } 1 \le i,j \le n\},
\end{aligned}$$

respectively. The dimension of the Schubert variety $X_w(E_\bullet)$ is $l(w)$, the number of inversions of w.

A *Schubert problem* asks for the number of flags $F_\bullet$ whose relative positions with respect to d given fixed flags $E^1_\bullet, \ldots, E^d_\bullet$ are given by the permutations $w^1, \ldots, w^d$. This question only makes sense when

$$X = X_{w^1}(E^1_\bullet) \cap \cdots \cap X_{w^d}(E^d_\bullet)$$

is 0-dimensional; that is, when $l(w^1) + \cdots + l(w^d) = \binom{n}{2}$. If $E^1_\bullet, \ldots, E^d_\bullet$ are sufficiently generic, the intersection X has a fixed number of points $c_{w^1 \ldots w^d}$ which only depends on the permutations $w^1, \ldots, w^d$.

This question is a fundamental one for several reasons; the numbers $c_{w^1 \ldots w^d}$ which answer it appear in another important context. The cycles $[X_w]$ corresponding to the Schubert varieties form a $\mathbb{Z}$-basis for the cohomology ring of the flag manifold $\mathcal{F}\ell_n$, and the numbers c_{uvw} are the multiplicative structure constants. (For this reason, if we know the answer to all Schubert problems with $d = 3$, we can easily obtain them for higher d.) The analogous structure constants in the Grassmannian are the Littlewood-Richardson coefficients, which are much better understood. For instance, even though the c_{uvw}s are known to be positive integers, it is a long standing open problem to find a combinatorial interpretation of them.

Billey and Vakil [6] showed that the permutation arrays of Eriksson and Linusson can be used to explicitly intersect Schubert varieties, and compute the numbers $c_{w^1 \ldots w^d}$.

Theorem 9.1. *(Billey-Vakil, [6]) Suppose that*

$$X = X_{w^1}(E^1_\bullet) \cap \cdots \cap X_{w^d}(E^d_\bullet)$$

is a 0*-dimensional and nonempty intersection, with $E^1_\bullet, \ldots, E^d_\bullet$ generic.*

1. *There exists a unique permutation array $P \subset [n]^{d+1}$, easily constructed from $w^1, \ldots, w^d$, such that*

$$\dim\big(E^1_{x_1} \cap \cdots \cap E^d_{x_d} \cap F_{x_{d+1}}\big) = rk\, P[x_1, \ldots, x_d, x_{d+1}],$$

   for all $F_\bullet \in X$ and all $1 \leq x_1, \ldots, x_{d+1} \leq n$.

2. *These equalities can be expressed as a system of determinantal equations in terms of the permutation array P and a vector $v_{a_1,\ldots,a_d}$ in each one-dimensional intersection $E_{a_1,\ldots,a_d} = E^1_{a_1} \cap \cdots \cap E^d_{a_d}$. This gives an explicit set of polynomial equations defining X.*

Theorem 9.1 highlights the importance of studying the line arrangements $\mathbf{E}_{n,d}$ determined by intersecting d generic complete flags in $\mathbb{C}^n$. In principle, if we are able to construct such a line arrangement, we can compute the structure constants c_{uvw} for any $u, v, w \in S_n$. (In practice, we still have to solve the system of polynomial equations, which is not easy for large n or for $d \geq 5$.) Let us make two observations in this direction.

9.1 Matroid genericity versus Schubert genericity.

We have been talking about the line arrangement $\mathbf{E}_{n,d}$ determined by a generic flag arrangement $E^1_\bullet, \ldots, E^d_\bullet$ in $\mathbb{C}^n$. We need to be careful, because we have given two different meanings to the word *generic*.

In Sections 3, 4 and 5, we showed that, if $E^1_\bullet, \ldots, E^d_\bullet$ are sufficiently generic, then the linear dependence relations in the line arrangement $\mathbf{E}_{n,d}$ are described by a fixed matroid $\mathcal{T}_{n,d}$. We call the flags *matroid-generic* if this is the case.

Recall that in the Schubert problem described by permutations $w^1, \ldots, w^d$ with $\sum l(w^i) = \binom{n}{2}$, the 0-dimensional intersection

$$X = X_{w^1}(E^1_\bullet) \cap \cdots \cap X_{w^d}(E^d_\bullet)$$

contains a fixed number of points $c_{w^1 \ldots w^d}$, provided that $E^1_\bullet, \ldots, E^d_\bullet$ are sufficiently generic. Let us say that n flags in $\mathbb{C}^d$ are *Schubert-generic* if they are sufficiently generic for any Schubert problem with that given n and d.

These notions depend only on the line arrangement $\mathbf{E}_{n,d}$. The line arrangement $\mathbf{E}_{n,d}$ is matroid-generic if its matroid is $\mathcal{T}_{n,d}$, and it is Schubert-generic if the equations of Theorem 9.1 give the correct number of solutions to every Schubert problem.

Our characterization of matroid-generic line arrangements (*i.e.*, our description of the matroid $\mathcal{T}_{n,d}$) does not tell us how to construct a Schubert-generic line arrangement. However, when $d = 3$ (which is the interesting case in the Schubert calculus), the cotransversality of the matroid $\mathcal{T}_{n,3}$ allows us to present such a line arrangement explicitly.

Proposition 9.2. *The* $\binom{n}{2}$ *path vectors of Theorem 6.4 are Schubert-generic.*

Proof. For each weighting L of the edges of the graph G_n with complex numbers, like the one shown in Figure 7, we can define the collection $V(L)$ of *path vectors* $v(L)_D = (v(L)_{D,1}, \ldots, v(L)_{D,n})$ as in Theorem 6.4: $v(L)_{D,i}$ is the sum of the weights of all paths from dot D to dot i on the bottom row of G_n.

Consider an arbitrary geometric representation V of $\mathcal{T}_{n,3}$ in $\mathbb{C}^n$. By means of a linear transformation, we can assume that the vectors assigned to the bottom row are the standard basis $e_1, \ldots, e_n$, in that order. Say D is any dot in the triangular array $T_{n,3}$, and E and F are the dots below it. Since D, E and F are dependent, and E and F are not, we can write $v_D = ev_E + fv_F$ for some $e, f \in \mathbb{C}$. Write the numbers e and f on the edges DE and DF of G_n. Do this for each dot D, and let L be the resulting weighting of the edges of G_n. Then the collection V is precisely the collection $V(L)$ of path vectors of L.

This shows that each matroid-generic line arrangement, *i.e.*, each geometric representation of $\mathcal{T}_{n,3}$, is given by the path vectors of a weighting of G_n. Among those matroid-generic line arrangements, the Schubert-generic ones form a Zariski open set, which will clearly include $V(L)$ for any sufficiently generic weighting L. This completes the proof. $\square$

A byproduct of proof of Proposition 9.2 is an interesting statement about the realization space of the matroid $\mathcal{T}_{n,3}$. Up to linear equivalence, every realization of $\mathcal{T}_{n,3}$ can be obtained from a weighting of the graph G_n.

Proposition 9.2 shows that when we plug the path vectors $V(L)$ into the polynomial equations of Theorem 9.1, and compute the intersection X, we will have $|X| = c_{uvw}$. The advantage of this point of view is that the equations are now written in terms of combinatorial objects, without any reference to an initial choice of flags.

Problem 9.3. *Interpret combinatorially the c_{uvw} solutions of the above system of equations, thereby obtaining a combinatorial interpretation for the structure constants c_{uvw}.*

Question 9.4. *Is a Schubert generic flag arrangement always matroid generic?*

Question 9.5. *Is a matroid generic flag arrangement always Schubert generic?*

9.2 A criterion for vanishing Schubert structure constants.

Consider the Schubert problem

$$X = X_{w^1}(E^1_\bullet) \cap \cdots \cap X_{w^d}(E^d_\bullet).$$

Let $P \in [n]^{d+1}$ be the permutation array which describes the dimensions $\dim(E^1_{x_1} \cap \cdots \cap E^d_{x_d} \cap F_{x_{d+1}})$ for any flag $F_\bullet \in X$. Let $P_1, \ldots, P_n$ be the n "floors" of P, corresponding to $F_1, \ldots, F_n$, respectively. Each one of them is itself a permutation array of shape $[n]^d$.

Billey and Vakil proposed a simple criterion which is very efficient in detecting that many Schubert structure constants are equal to zero.

Proposition 9.6. *(Billey-Vakil, [6]) If P_n is not the transversal permutation array, then $X = \emptyset$ and $c_{w^1 \ldots w^d} = 0$.*

Knowing the structure of the matroid $\mathcal{T}_{n,d}$, we can strengthen this criterion as follows.

Proposition 9.7. *Suppose P_n is the transversal permutation array, and identify it with the set $T_{n,d}$. If, for some k, the rank of $P_k \cap P_n$ in $\mathcal{T}_{n,d}$ is greater than k, then $X = \emptyset$ and $c_{w^1 \ldots w^d} = 0$.*

Proof. Each dot in P_n corresponds to a one-dimensional intersection of the form $E^1_{x_1} \cap \cdots \cap E^d_{x_d}$. Therefore, each dot in $P_k \cap P_n$ corresponds to a line that F_k is supposed to contain if $F_\bullet$ is a solution to the Schubert problem. The rank of $P_k \cap P_n$ is the dimension of the subspace spanned by those lines; if $F_\bullet$ exists, that dimension must be at most k. □

Let us see how to apply Proposition 9.7 in a couple of examples. Following the method of [6], the permutations $u = v = w = 213$ in S_3 give rise to the four-dimensional permutation array consisting of the dots $(3, 3, 1, 1)$,

$(1,3,3,2)$, $(3,1,3,2)$, $(3,3,1,2)$, $(1,3,3,3)$, $(2,2,3,3)$, $(2,3,2,3)$, $(3,1,3,3)$, $(3,2,2,3)$, and $(3,3,1,3)$. We follow [13, 44] in representing it as follows:

		1

		3
3		1

		3
	3	2
3	2	1

The three boards shown represent the three-dimensional floors P_1, P_2, and P_3 of P, form left to right. In each one of them, a dot in cell (i,j,k) is represented in two dimensions by a number k in cell (i,j).

It takes some practice to interpret these tables; but once one is used to them, it is very easy to proceed. We simply notice that $P_2 \cap P_3$ is a set of rank 3 in the matroid $\mathcal{T}_{3,3}$, while P_2 has rank 2 as a permutation array; we conclude that $c_{213,213,213} = 0$. For $n = 3$, this is the only vanishing c_{uvw} which is not explained by Proposition 9.6. In this example, the vanishing of c_{uvw} can also be seen by comparing the leading terms of the corresponding Schubert polynomials.

For a larger example, let $u = 2134, v = 3142, w = 2314$. Notice that $l(u) + l(v) + l(w) = 6$. The permutation array we obtain is

	4		

			4
	4		1

			4
	4		3
4	3		1

			4
		4	3
	4	3	2
4	3	2	1

and, since $P_3 \cap P_4$ has rank 4 in $\mathcal{T}_{4,3}$, we see that $c_{2134,3142,2314} = 0$.

Knutson [21], Lascoux and Schützenberger [24], Purbhoo [36], and Purbhoo and Sottile [37] have developed other methods for detecting the vanishing of Schubert structure constants. In comparing these methods for small values of n, we have found Proposition 9.7 to be quicker and simpler to observe, but we have not been able to verify our technique as far as Purbhoo; he has the best technique thus far, detecting all zeros for $n \leq 7$.

Here is an example where our method allows us to "observe" a zero coefficient that Knutson [21, Fact 2.4] claims does not follow from his technique of descent cycling. Let $u = 231645$, $v = 231645$, $w = 326154$, then the unique permutation array determined by these three permutations is:

$$
\begin{array}{|c|c|c|c|c|c|}\hline
 & & & & & \\\hline
 & & & & & \\\hline
 & & & & & \\\hline
 & & & & & \\\hline
 & & & & 4 & \\\hline
 & & & & & \\\hline
\end{array}
\quad
\begin{array}{|c|c|c|c|c|c|}\hline
 & & & & & \\\hline
 & & & & & \\\hline
 & & & & & \\\hline
 & & & & 5 & \\\hline
 & & & 5 & 4 & \\\hline
 & & & & & \\\hline
\end{array}
\quad
\begin{array}{|c|c|c|c|c|c|}\hline
 & & & & & \\\hline
 & & & & & \\\hline
 & & & & & \\\hline
 & & & & 5 & \\\hline
 & & & 5 & 4 & \\\hline
 & & & & & 1 \\\hline
\end{array}
\tag{1}
$$

$$
\begin{array}{|c|c|c|c|c|c|}\hline
 & & & & & 6 \\\hline
 & & & & & \\\hline
 & & & & & \\\hline
 & & & & 5 & \\\hline
 & & & 5 & 4 & \\\hline
6 & & & & & 1 \\\hline
\end{array}
\quad
\begin{array}{|c|c|c|c|c|c|}\hline
 & & & & & 6 \\\hline
 & & & & & \\\hline
 & & & & 6 & 5 \\\hline
 & & & 6 & 5 & 4 \\\hline
 & & 6 & 5 & 4 & 2 \\\hline
6 & & 5 & 4 & 2 & 1 \\\hline
\end{array}
\quad
\begin{array}{|c|c|c|c|c|c|}\hline
 & & & & & 6 \\\hline
 & & & & 6 & 5 \\\hline
 & & & 6 & 5 & 4 \\\hline
 & & 6 & 5 & 4 & 3 \\\hline
 & 6 & 5 & 4 & 3 & 2 \\\hline
6 & 5 & 4 & 3 & 2 & 1 \\\hline
\end{array}
\tag{2}
$$

By Theorems 4.1 and 5.1, an independent set on the rank 6 board is determined by the circled points

$$\{(1,6,6),(3,5,5),(4,4,5),(5,3,5),(6,1,6),(6,6,1)\}.$$

In the rank 4 board, we have the points $(1,6,6),(6,1,6),(6,6,1)$ from this basis along with $(4,5,5),(5,4,5),(5,5,4)$ which span a two dimensional space in the span of $(3,5,5),(4,4,5),(5,3,5)$. Therefore, the 4 dimensional board cannot be satisfied by vectors in a space of dimension less than 5. Hence $c_{uvw}=0$.

Proposition 9.7 is only the very first observation that we can make from our understanding of the structure of $\mathcal{T}_{n,d}$. Our argument can be easily fine-tuned to explain all vanishing Schubert structure constants with $n \leq 5$. A systematic way of doing this in general would be very desirable.

10 Future directions.

We invite our readers to pursue some further directions of study suggested by the results in this paper. Here they are, in order of appearance.

- Theorem 6.2 generalizes to rhombus tilings of any region in the triangular lattice, or domino tilings of any region in the square lattice. If R is a region with more upward than downward triangles (or more black than white squares), let $\mathcal{B}$ be the sets of holes such that the remaining holey region R has a rhombus tiling (or a domino tiling). Then $\mathcal{B}$ is the set of bases of a matroid M_R. Are there any other regions R for which the matroid M_R has a nice geometric interpretation? A good

Figure 12: A fool's diamond.

candidate, suggested by Jim Propp, is what he calls the *fool's diamond* [35], shown in Figure 12.

- Subdivisions of $\Delta_{n-1} \times \Delta_{d-1}$, or equivalently tropical oriented matroids, appear in many different contexts. A detailed investigation of these objects promises to become a useful tool. Aside from their intrinsic interest, Conjecture 7.1 and Question 8.3 should help us develop this tropical oriented matroid theory.

- We still do not have a solid understanding of the relationship between two of the main subjects of our paper: the geometry of d flags in $\mathbb{C}^n$ and the triangulations of $\Delta_{n-1} \times \Delta_{d-1}$. We have shown that some aspects of the geometric information of the flags (the combinatorics of the line arrangement they determine, and the vanishing of many Schubert structure constants) are described in a small set of tiles of the triangulations (the n "pure" tiles). Can we use the complete triangulations and fine mixed subdivisions to understand more subtle geometric questions about flags? Does the geometry of flags tell us something new about triangulations of products of simplices, and their multiple appearances in tropical geometry, optimization, and other subjects?

- In particular, do the triangulations of $\Delta_{n-1} \times \Delta_{d-1}$ play a role in the Schubert calculus of the flag manifold $\mathcal{F}\ell_n$? Is this point of view related to Knutson, Tao, and Woodward's use of puzzles [22, 23] in the Grassmannian Schubert calculus? Readers familiar with puzzles may have noticed the similarities and the differences between them and lozenge tilings of triangles.

- Problem 9.3 is a promising way of attacking the long-standing open problem of interpreting c_{uvw} combinatorially.

- Questions 9.4 and 9.5 remain open. Is a Schubert generic flag arrangement always matroid generic? Is a matroid generic flag arrangement always Schubert generic?

- Proposition 9.7 is just the first consequence of the matroid $\mathcal{T}_{n,d}$ on the vanishing of the Schubert structure constants. This argument can be extended in many ways to explain why other c_{uvw}s are equal to 0. A systematic way of doing this would be desirable, and seems within reach at least for $n \leq 7$.

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