

Combinatorics of the cosmohedron



Federico Ardila-Mantilla*

Nima Arkani-Hamed[†]

Carolina Figueiredo[‡]

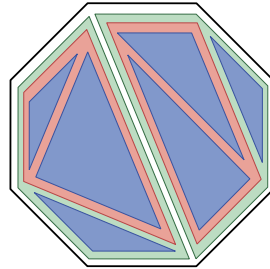
Francisco Vazo[§]

Abstract

The cosmohedron was recently proposed as a polytope underlying the cosmological wavefunction for $\text{Tr}(\Phi^3)$ theory. Its faces were conjectured to be in bijection with Matryoshkas, which are obtained from a subdivision of a polygon by sequentially wrapping groups of polygons into larger polygons. In this paper we prove the correctness of this construction, and elucidate its combinatorial structure. Cosmohedra generalize to a wider class of \mathcal{X} in Y polytopes, where we chisel a polytope from the family \mathcal{X} at each vertex of a polytope Y . We sketch a new application of these chiseled polytopes to the physics of ultraviolet divergences in loop-integrated Feynman amplitudes.

1 Introduction

It has been a long standing question in cosmology to find a geometric object that encodes the cosmological wavefunction of $\text{Tr}(\phi^3)$ theory. The combinatorics of this wavefunction is governed by certain nesting subdivisions of a polygon that we call *Matryoshkas*.



Arkani-Hamed, Figueiredo, and Vazão [AHFV25] proposed a construction of the *cosmohedron* as a solution to this cosmological question. The goal of this article is to prove the correctness of that subtle construction. Along the way we recover and discover numerous physical, combinatorial, polyhedral, and enumerative properties of Matryoshkas and cosmohedra, that we briefly outline in this introduction.

*Queen Mary University of London and San Francisco State University; f.ardila@qmul.ac.uk.

[†]Institute for Advanced Study; arkani@ias.edu.

[‡]Jadwin Hall, Princeton University; cfigueiredo@princeton.edu.

[§]Institute for Advanced Study and University of Pennsylvania; fvvazao@ias.edu.

1.1 Combinatorial properties of the cosmohedron

The combinatorial structure of the cosmological wavefunction of $\text{Tr}(\phi^3)$ theory is described by the *Matryoshkas* of an $(n + 2)$ -gon \diamond_{n+2} , so these are the objects we wish to model geometrically in the cosmohedron. We call them Matryoshkas because their nesting structure is reminiscent of – although much richer than – that of the Matryoshka dolls created by S. Malyutin and V. Zvyozdochkin in Russia in the 1890s.

One way of constructing a Matryoshka, “from the inside out” is to choose a subdivision of \diamond_{n+2} , and then repeatedly choose a group of adjacent outermost polygons and wrapping them into a larger polygon engulfing them. Alternatively, “from the outside in”, a Matryoshka is obtained recursively by choosing a subdivision of \diamond_{n+2} , and then placing a Matryoshka inside each of its polygons.

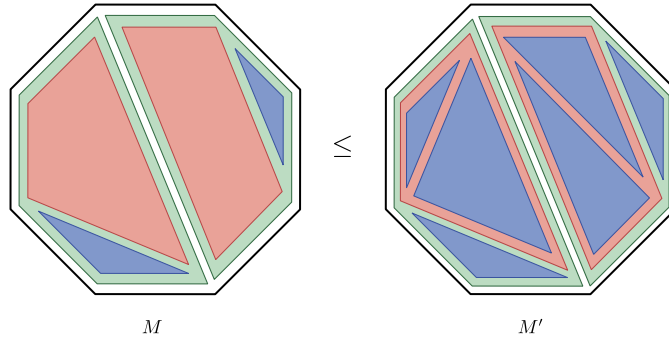


Figure 1: An octagonal Matryoshka M and a maximal Matryoshka M' containing it.

Naturally, we can think of a Matryoshka as a set of polygons. This gives a partial order on the set of Matryoshkas on \diamond_{n+2} by *containment*: we have $M \leq M'$ if every polygon of M is also in M' .

Arkani-Hamed, Figueiredo, and Vazão introduced an $(n - 1)$ -dimensional polytope called the *cosmohedron* to encode this combinatorial structure; see Definition 4.15. Our main result is a proof of the following statement from [AHFV25].

Theorem 1.1. *The face lattice of the $(n - 1)$ -cosmohedron is anti-isomorphic to the poset of Matryoshkas of an $(n + 2)$ -gon.*

1.2 Polyhedral properties of the cosmohedron

The search for a cosmohedron that models the combinatorics of Matryoshkas is analogous, and in fact closely related, to earlier constructions of polyhedra with a desired combinatorial structure. An important motivating example is the *associahedron*, whose faces are in bijection with the plane trees of a given size.

The cosmohedron is obtained from the associahedron through a careful chiseling procedure: at each vertex a_T corresponding to the binary tree T , we chisel a copy of the *bracket associahedron* Br_T . The chiselings at the different vertices interact with each

other, so they must be carried out very precisely. We must place the right realization of each Br_T in the correct position, angle, and size, so that the different bracket associahedra glue correctly into one polytope with the desired combinatorics. The 3-cosmohedron is shown in Figure 2. The precise construction is given in Section 4.

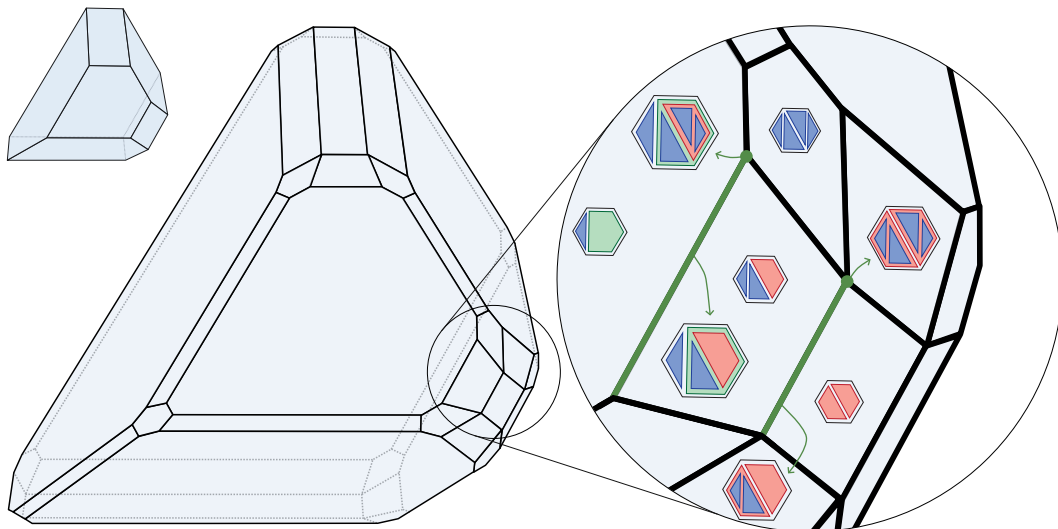


Figure 2: The 3-cosmohedron as a blowup of the 3-associiahedron, and the Matryoshkas labeling some of its faces.

Arkani–Hamed, Figueiredo, and Vazão [AHFV25] had described the cosmohedron AFV_{n-1} inside the intersection of an $(n-1)$ -dimensional plane with the higher-dimensional positive orthant in $\mathbb{R}_{\geq 0}^{(n+2)(n-1)/2}$, imposing an additional family of inequalities; see Definition 4.15. This embedding gives the polytope a beautifully simple inequality description, but it is not well suited to describe the normal fan – an essential ingredient in proving the combinatorial structure.

We propose a different construction Cosmo_{n-1} of the cosmohedron (a hyperplane in) \mathbb{R}^n which is closer in spirit to the family of generalized permutahedra. We begin by building its normal fan CosmoFan_{n-1} in \mathbb{R}^n/\mathbb{R} , and proving that it has the correct combinatorial structure of Matryoshkas; we do this in Definition 2.3 and Theorem 3.6. Next we construct explicit coordinates for the vertices of Cosmo_{n-1} , and prove that its normal fan is CosmoFan_{n-1} in Theorem 4.11. Since we know the rays of the fan, we can readily use them to compute the inequality description of Cosmo_{n-1} in Theorem 4.13. Finally, with inequality descriptions at hand for both of them, we find a linear isomorphism between Cosmo_{n-1} in \mathbb{R}^n and AFV_{n-1} in $\mathbb{R}^{(n+2)(n-1)/2}$ in Theorem 4.16. This proves the combinatorial structure of the cosmohedron AFV_{n-1} as conjectured in [AHFV25].

1.3 Enumerative properties of the cosmohedron

In Section 5 we turn to enumerative questions. The facets of the cosmohedron are in bijection with the non-trivial subdivisions of \diamond_{n+2} , which are enumerated by the Hipparchus–Schröder numbers. This sequence is well understood, and is in fact one of the earliest combinatorial sequences, dating to the second century BC; for an interesting historical account, see [Sta99, Sta97].

The vertices correspond to the maximal Matryoshkas; we prove that they are enumerated by sequence A177384 in the OEIS. We give a recursive formula for the sequence and a simple polynomial differential equation for the power series:

Theorem 1.2. *The generating function $M(x) = x^2 + 2x^3 + 10x^4 + 72x^5 + \dots$ for the number of maximal Matryoshkas is characterized by the polynomial differential equation*

$$M(x) = x^2 + M(x)M'(x).$$

Thus Matryoshkal numbers form one of the simplest sequences in the notoriously difficult family of *D-algebraic power series* [Mel21]. For instance, we do not know how to prove the asymptotic behavior of this sequence; see Conjecture 5.7.

We also give an intriguing description for the f -vectors of cosmohedra: their generating function equals its own compositional inverse up to a simple transformation. Consider the f -polynomials of the cosmohedral fans $f_n(t)$ as well as the simple transformation $g_n(t) = ((1+t)f_n(t) - 1)/t$. The coefficients of $g_n(t)$ are obtained by adding consecutive coefficients of $f_n(t)$:

$$\begin{aligned} f_1(t) &= 1, f_2(t) = 1 + 2t, f_3(t) = 1 + 10t + 10t^2, f_4(t) = 1 + 44t + 114t^2 + 72t^3, \dots \\ g_1(t) &= 1, g_2(t) = 3 + 2t, g_3(t) = 11 + 20t + 10t^2, g_4(t) = 45 + 158t + 186t^2 + 72t^3, \dots \end{aligned}$$

Theorem 1.3. *The f -polynomials of cosmohedra form the unique sequence $\{f_n(t)\}_{n \geq 1}$ such that the power series in $(\mathbb{R}[t])[x]$*

$$\begin{aligned} x - f_1(t)x^2 - f_2(t)x^3 - f_3(t)x^4 - \dots \\ x + g_1(t)x^2 + g_2(t)x^3 + g_3(t)x^4 + \dots \end{aligned}$$

are compositional inverses, where $g_n(t) = ((1+t)f_n(t) - 1)/t$ for $n \geq 1$.

We also give a strikingly similar formula for the f -vector of the correlatron.

1.4 Elusive qualities of the cosmohedron

Cosmohedra are related to several polytopes in the mathematical literature, some of which are known to be relevant in physical applications. These include permutahedra [BS13, Sch13], associahedra [CSZ15, Lod04, Sta63, Tam54], bracket associahedra [BCL⁺22, LA22] (which are special cases of graph associahedra [CD06, Dev09, Pos09]), permutoassociahedra [CL23, Kap93], and permutonestohedra [Gai15], among others.

In Section 6 we explain that the cosmohedron is more subtle than its predecessors in several ways: it is not very symmetric, it is not simple, it depends very delicately on

the chiseling, we do not know how to express it as a deformation of an easy polytope, we do not know how to express it as a Minkowski sum of easy polytopes, and its faces factor combinatorially but not geometrically. For these reasons, proving its combinatorial structure requires new ideas and contributions, which we now describe.

1.5 Cosmological properties of the cosmohedron

Let us close this introduction by briefly explain the physical origins of this project. The content of this section is not essential to the mathematical understanding of this paper, but it is essential to the discovery of these mathematical objects, and it gives rise to many interesting new directions.

The past decade has seen the discovery of startling and deep new connections between fundamental physics and new mathematical structures in combinatorics and geometry. The story began with the most basic and fundamental physical processes in nature—the collision and scattering of elementary particles. Because of quantum mechanics, the outcome of any given scattering process is uncertain, only the probability for different states can be predicted. Given an initial state i and a final state f , the probability is $|\mathcal{A}_{i,f}|^2$ where $\mathcal{A}_{i,f}$ is the *scattering amplitude*. The textbook method for computing these amplitudes is to sum over *Feynman diagrams* representing the different ways in which the collision events can take place in space and time.

The connection to combinatorics and geometry can most easily and vividly be illustrated in an especially useful toy model, the so-called $Tr(\phi^3)$ theory, for “colored” scalar particles represented by $N \times N$ square matrices. In the simplest limit (“tree-level”), the amplitude turns into the sum over all planar trivalent (known in physics as *cubic*) tree diagrams, or what is the same, a sum over the triangulations of an n -gon.

A priori to a physicist, there is no obvious combinatorics here: we simply have a collection of diagrams, that have to be collected and summed over, in no particular order, to arrive at the final result. But a combinatorialist knows that there is something very special about this particular collection of objects: the triangulations of an n -gon are captured by the vertices of a polytope, the associahedron.

This link turns out to be the tip of a large iceberg of much larger significance to physics beyond “merely” the combinatorics of diagrams. A particular embedding of the associahedron allows us to extract mathematical expressions for the scattering amplitude from the geometry [AHBHY18a]. This particular embedding makes it natural to think of the associahedron as a Minkowski sum of the simplest possible pieces – a collection of simplices – and quite remarkably, this fact is intimately related to (and indeed lets us naturally discover from the bottom-up) the generalization from particle scattering to scattering processes in string theory. Over the past few years, these developments have led to the discovery of a number of surprising connections between seemingly totally different physical theories [AHCD⁺24], and almost as an incidental byproduct, have provided new ways of simplifying enormously complicated Feynman diagram calculations, sometimes into a few lines.

The connection to physics also led to the discovery of new mathematical objects. For instance, the associahedron captures all tree diagrams, but what about diagrams

that can be drawn on surfaces with arbitrary topology? The consistency of the physical picture demands that geometric objects encoding such diagrams must exist, and they have indeed been described in the physics literature [AHFS⁺25].

This emerging connection between physics and combinatorics is suggesting a radical new picture for the laws of physics, where the notion of particle evolution in spacetime is not primary. Instead, the unifying combinatorial-geometric objects that control how all spacetime processes are “glued together” play a more fundamental role. This dovetails nicely with an expectation, widely shared by most theoretical physicists, that the notion of spacetime can not be fundamental, but must eventually be supplanted by more elementary and abstract ideas. The new connection to combinatorics suggests a concrete strategy for pursuing this vision.

Nowhere is the need for a new picture of physics supplanting spacetime more urgent than in cosmology. Amongst other things, the notion of time itself breaks down at the big bang singularity, so the standard picture of time evolution must likely be abandoned.

Cosmology also offers us a rich set of physical observables to compute, and in principle measure, that turn out to be interesting generalizations of scattering amplitudes. It is widely believed that the very early history of the universe involved a period of exponentially rapid expansion known as *inflation*, and that quantum fluctuations in this early inflationary phase were blown up to enormous scales, even larger than our observable universe, giving rise to all the “clumpiness” in matter and all the interesting structures we see in the universe today. The details of these quantum fluctuations are reflected in the statistics of the distribution of mass and energy in the universe today, and are known as *cosmological correlation functions*. The cosmological correlators produced during inflation are computed in much the same way as particle scattering amplitudes, drawing diagrams for all possible ways the correlations could have been produced. But there is an important difference: while for scattering processes a single diagram represents a single elementary process, in cosmology, even a single diagram has much more structure, precisely having to do with the different time-ordering of “histories” a given diagram can represent.

If we return to the toy $\text{Tr}(\phi^3)$ theory, to compute a cosmological correlator, a given diagram must be further decorated, by all the possible *bracketings* of the graph or equivalently, what we will call the possible *Matryoshkas* associated with the corresponding triangulation. This arises in a very natural way when solving the Schrödinger equation $H\Psi = E\Psi$ to compute the wavefunction $\Psi(\phi)$ for ϕ , from which we determine probabilities and correlation functions. The Schrödinger equation for $\Psi(\phi) = \exp[\psi(\phi)]$ can be solved in perturbation theory. The exponential relating $\Psi(\phi)$ to $\psi(\phi)$ reflects the familiar exponential relationship between all graphs and connected graphs. Every connected cubic diagram with n external legs, where the external legs carry spatial momentum \vec{k}_i , makes a contribution to the coefficient of $\phi(\vec{k}_1) \cdots \phi(\vec{k}_n)$ in the Taylor expansion of $\psi(\phi)$ in powers of ϕ . The contribution of a diagram to ψ depends on “energies” entering each subgraph – this is the total sum of the magnitude of the momenta entering the subgraph. In Figure 3, these energies are given by perimeters of subpolygons.

Written in terms of ψ , the Schrödinger equation turns into a simple recursion relation:

the contribution from a given graph can be computed by first writing a factor of the inverse of the sum over all energies on the vertices, and then summing over all the ways of deleting an edge from the graph, and adding the energy of the edge to the vertices it connects to in the smaller graph with the edge deleted. We can run this recursion until there are no more edges to be cut, and every term encountered corresponds to a bracketing of the graph – where to each bracket subgraph we associate the factor given by the total energy entering that subgraph. Phrased in the language of the triangulations, each bracket corresponds to a sub-polygon in the triangulation, to which we associate a factor given by its inverse perimeter. The terms contributing to the wavefunction are then associated with a maximal collection of nested sub-polygons in the triangulation, corresponding to a Matryoshka; see Figure 3.

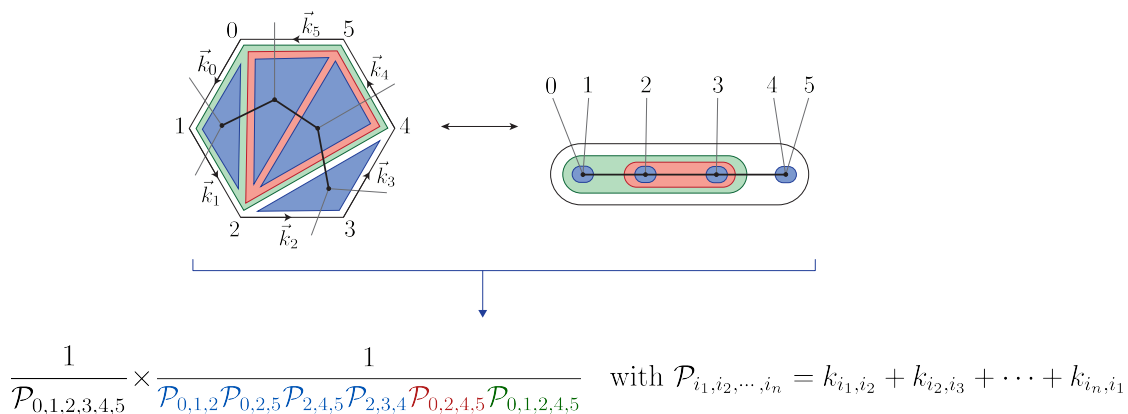


Figure 3: The contribution of a tree graph (or equivalently a triangulation) to the wavefunction is determined by the bracketings of the graph (or equivalently the Matryoshkas of the triangulation). Here $k_{i,j}$ denotes the length between vertices i and j , which corresponds to the length of the momenta flowing through the edges of the graph, and \mathcal{P} stands for the perimeter of the subpolygons entering the Matryoshka.

Given the close connection between scattering amplitudes and cosmological correlators, physics suggested that some analog of the associahedron must exist, capturing not the combinatorics of all triangulations, but the factorially larger combinatorics of all possible Matryoshkas. This was the motivation leading to the discovery and definition of cosmohedra, which are polytopes precisely realizing this picture.

Our main goal in this paper is to better understand the combinatorics and geometry of cosmohedra, provide proofs of many of their claimed properties, and also describe them in a way that makes more natural contact with similar objects that have recently been studied in combinatorics.

1.6 Outlook: \mathcal{X} in Y polytopes and related constructions in physics

Apart from its intrinsic mathematical interest, our treatment makes it clear that cosmohedra belong to a wider class of objects whose relevance to physics should extend far beyond

cosmology. We will see that cosmohedra generalize to what we can call \mathcal{X} in Y polytopes, where we begin with a polytope Y and “chisel” it in a way controlled by a family \mathcal{X} of polytopes to get a new *chiseled polytope*. At the level of fans, the normal fan of a (simple) polytope Y is canonically refined using the normal fans of the polytopes in \mathcal{X} . We expect these \mathcal{X} in Y polytopes to be relevant to many settings in physics, where, in addition to summing over all diagrams, we must keep track of various properties of subgraphs in each diagram.

An especially important example of this arises in the context of scattering amplitudes beyond “tree” level. In diagrams with loops, the internal momenta are not determined uniquely, and must be integrated over. This is associated with a rich pattern of possible divergences when loop momenta become large (“ultraviolet”) or small (“infrared”). For an individual Feynman diagram, these divergences are controlled by the Newton polytopes of certain graph polynomials – the so called *Symanzik polynomials* – associated with the graph. In [AHCD⁺24], all Feynman diagrams are naturally realized as cones glued together into a complete “Feynman fan”, but this raises a question posed in [AHCD⁺24]: can the combinatorics of the “Symanzik polytopes” associated with any graph be encoded in some way inside the full Feynman fan that glues all diagrams together? This is exactly what \mathcal{X} in Y polytopes allow us to do. In the Appendix, we sketch this construction in the first interesting case: the Symanzik \mathcal{U} -polynomials at one-loop level in the $\text{Tr}(\phi^3)$ theory.

2 Matryoshkas, bracketed trees, and the cosmohedral fan

2.1 Matryoshkas

Consider a convex $(n+2)$ -gon P . A *subdivision* \mathcal{S} is a set of sub-polygons whose union is P and whose interiors are pairwise disjoint. A subdivision \mathcal{S} can be naturally identified with the corresponding set \mathcal{C} of non-crossing diagonals of P . We say \mathcal{S} *refines* \mathcal{S}' , and write $\mathcal{S} \geq \mathcal{S}'$, if the following three equivalent conditions hold: every polygon in \mathcal{S} is contained in a polygon in \mathcal{S}' , every polygon in \mathcal{S}' is a union of polygons in \mathcal{S} , and the corresponding sets of chords satisfy $\mathcal{C} \supseteq \mathcal{C}'$.

A *Matryoshka* or *Russian doll* M is a set of sub-polygons of P such that for any non-minimal polygon X in M , the maximal subpolygons $(M_{<X})_{\max}$ of X that are in the Matryoshka M subdivide X . In particular, the set M_{\min} of minimal polygons in M form the *underlying subdivision* of M . We say M *refines* M' , and write $M \geq M'$, if the sets of polygons satisfy $M \supseteq M'$.

For example, Figure 4 shows two maximal Matryoshkas M_1 and M_2 on a hexagon. Removing a quadrilateral from M_1 gives a Matryoshka smaller than it. Removing a single triangle from M_1 does not give a non-Matryoshka, because the corresponding quadrilateral is not subdivided by its maximal subpolygons.

Recursively a Matryoshka is either:

- $M = \{P\}$, the full polygon, or
- $M = \mathcal{S} \cup \bigcup_{S \in \mathcal{S}} M_S$: a subdivision \mathcal{S} of P and a Matryoshka M_S on each part S of \mathcal{S} .

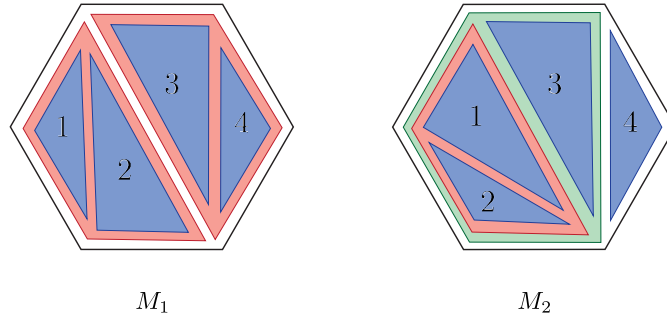


Figure 4: Two maximal Matryoshkas M_1 and M_2 on a hexagon.

Definition 2.1. *The poset of Matryoshkas Mat_{n-1} is the partially ordered set of Matryoshkas on the $(n+2)$ -gon ordered by containment, with an additional maximum element $\hat{1}$.*

2.2 The cosmohedral fan in terms of Matryoshkas

Let us give the vertices of our $(n+2)$ -gon $P = \diamond_{n+2}$ the labels $0, 1, \dots, n+1$ counterclockwise, with the edge e joining 0 and $n+1$ placed horizontally at the top.

Consider a triangulation (or subdivision) \mathcal{S} of P .

Triangle (or polygon) labels. We have a canonical labeling of the n triangles of \mathcal{S} with the vertex labels $1, \dots, n$: We give the label v to the first triangle of \mathcal{S} that one encounters on a straight path from vertex v to edge e . (In a subdivision, a k -gon gets $k-2$ such labels.)

Diagonal labels. Each diagonal of \mathcal{S} separates two triangles labeled v and w (or polygons labeled V and W), with triangle v between triangle w and edge e ; we give that diagonal the label vw^\perp (or VW^\perp).

Now consider a Matryoshka M of P , with underlying triangulation (or subdivision) \mathcal{S} .

Polygon labels. Each diagonal vw^\perp is contained inside a unique smallest polygon of M , which we denote $\diamond_M(vw^\perp)$. When M is maximal, this is a bijection between the diagonals of \mathcal{S} and the non-minimal polygons of M .

Containment poset. The diagonals vw^\perp are the elements of the *containment poset* τ of M , whose order relations are given by containment of the corresponding polygons: $ij^\perp \geq kl^\perp$ if $\diamond_M(ij^\perp) \supseteq \diamond_M(kl^\perp)$. This poset is a tree, which is binary if M is maximal.

Matryoshkal cone. To the Matryoshka M we assign the cone

$$\text{cone}(M) = \{x \in \mathbb{R}^n : x_i - x_j \geq x_k - x_l \geq 0 \text{ when } \diamond_M(ij^\perp) \supseteq \diamond_M(kl^\perp)\} \quad (1)$$

naturally associated to the containment poset.

Example 2.2. *In the Matryoshka M_1 of Figure 5 we have $\diamond(21^\perp) \subset \diamond(32^\perp)$ and $\diamond(34^\perp) \subset \diamond(32^\perp)$, so $\text{cone}(M) = \{x \in \mathbb{R}^4 : x_3 - x_2 \geq \{x_2 - x_1, x_3 - x_4\} \geq 0\}$.*

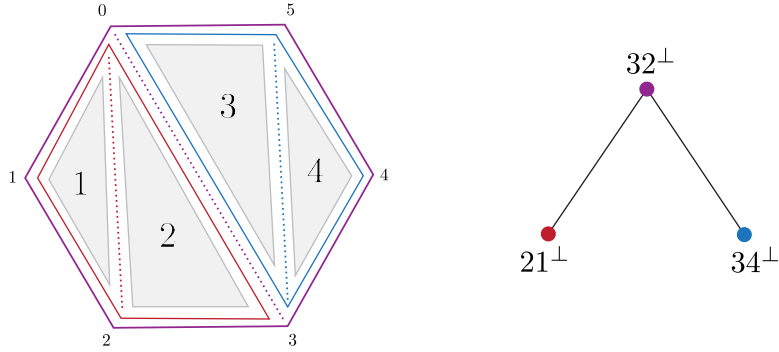


Figure 5: The Matryoshka M_1 and its containment poset τ_1 .

Definition 2.3. *The cosmohehedral fan is*

$$\text{CosmoFan}_{n-1} = \{\text{cone}(M) : M \text{ is a Matryoshka of the } (n+2)\text{-gon}\}.$$

We will prove in Section 3 that this is indeed a fan, and that it has the desired combinatorial structure:

Theorem 2.4. *The cosmohehedral fan is a complete fan in \mathbb{R}^n . The map $M \mapsto \text{cone}(M)$ is an order-preserving bijection between the poset Mat_{n-1} of Matryoshkas of an $(n+2)$ -gon and the face lattice $L(\text{CosmoFan}_{n-1})$ of the cosmohehedral fan.*

2.3 Bracketed trees and the cosmohehedral fan

Let us give a different combinatorial description of Matryoshkas, which gives rise to a description of the cosmohehedral fan in terms of *bracketed trees*.

2.3.1 Preliminaries: plane trees, triangulations, and the associahedral fan.

A *plane tree* is a rooted tree where the children of each node are linearly ordered from left to right. The plane trees on $n+1$ leaves form a poset Tree_{n-1} where $T \leq T'$ if T can be obtained from T' by contracting edges. We include an additional maximum element $\hat{1}$ that makes this poset into a lattice.

The coatoms of this poset are the *binary trees*, which have n internal nodes and $n+1$ leaves. It is sometimes convenient to erase the leaves of a binary tree, obtaining a plane tree with n vertices.

If \mathcal{S} is a subdivision of \diamond_{n+2} then the *dual graph* $T(\mathcal{S})$ has a vertex for each polygon of \mathcal{S} and an edge between each pair of polygons that share an edge. This is a tree that we root at the polygon adjacent to the top edge e . Figure 9 shows the binary trees dual to the subdivisions of Matryoshkas M_1 and M_2 .

If \mathcal{S} is a triangulation then $T(\mathcal{S})$ is a binary tree, and this is a bijection between the triangulations of \diamond_{n+2} and the trees in Tree_{n-1} ; the number of such objects is the Catalan number $C_n = \frac{1}{n+1} \binom{2n}{n}$.

The internal vertices of a plane tree T with $n + 1$ leaves are naturally labeled with sets that form a partition of $[n]$: we give the gaps between the leaves the labels $1, 2, \dots, n$ from left to right, and label each internal node of T with the gap(s) that are visible from it. We regard T as the Hasse diagram of a pre-poset on $[n]$ whose root is the maximum element, and define

$$\text{cone}(T) = \{x \in \mathbb{R}^n / \mathbb{R}\mathbf{1} : x_i \geq x_j \text{ for each } i \geq j \text{ in } T_{\text{int}}\}$$

where $\mathbf{1} = (1, \dots, 1)$. In particular $x_i = x_j$ in $\text{cone}(T)$ whenever i and j label the same vertex of T .

Definition 2.5. *The associahedral fan is*

$$\text{AssocFan}_{n-1} = \{\text{cone}(T) : T \text{ is a plane tree on } n + 1 \text{ leaves}\}.$$

Figure 6 shows the associahedral fan AssocFan_3 .

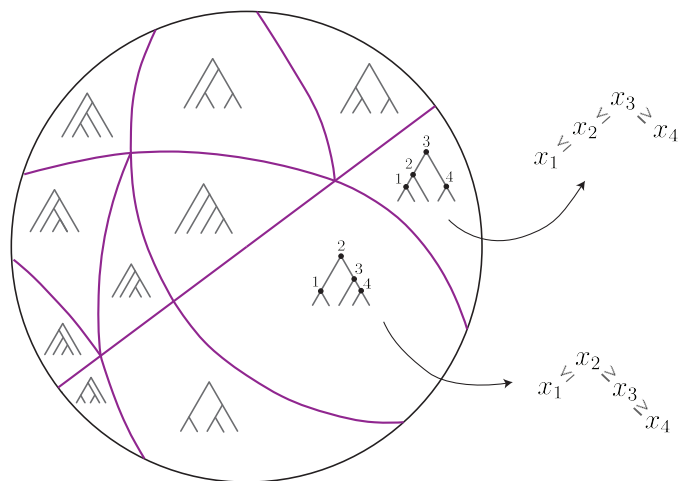


Figure 6: The three-dimensional associahedral fan.

Theorem 2.6. [Lod04] *The associahedral fan AssocFan_{n-1} is a complete fan in \mathbb{R}^n . The map $T \mapsto \text{cone}(T)$ is an order-preserving bijection between the poset Tree_{n-1} of plane trees with $n + 1$ leaves and the face lattice $L(\text{AssocFan}_{n-1})$ of the associahedral fan.*

2.3.2 Preliminaries: graph bracketings and bracket associahedral fans.

Let $H = (V, E)$ be a connected graph. A *bracket* of H is a connected subgraph of H with at least one edge. We identify it with its set of edges. A *bracketing* of H is a collection B of brackets such that any two brackets are either nested (one is contained in the other) or disjoint (they have no edges or vertices in common). We require every bracketing to contain the full graph H .

The *poset of bracketings* $\text{Br}(H)$ is the set of bracketings of H ordered by containment, with an additional maximum element $\hat{1}$.

In a bracketing B , each edge e is contained in a unique smallest bracket of B , which we denote $\text{bracket}_B(e)$ or simply $\text{bracket}(e)$. This defines a pre-poset tree $\text{tree}(B)$ on the edge set E , where $e \geq_B f$ if e is *outermore* than f in the sense that $\text{bracket}(e) \supseteq \text{bracket}(f)$; or equivalently, every bracket containing e also contains f . Because every bracket is contained in a unique smallest bracket, the poset $\text{tree}(B)$ is a tree rooted at a unique maximal element. We obtain a cone

$$\text{cone}_H(B) = \{y \in \mathbb{R}^{E(H)} : y_e \geq y_f \text{ for each } e \geq_B f\}.$$

Its dimension is $|B|$.

Definition 2.7. *The bracket associahedral fan of a graph H is*

$$\text{BrAssocFan}(H) = \{\text{cone}_H(B) : B \text{ is a bracketing of } H\}$$

Figure 7 illustrates the bracket associahedral fan of a path of length 4 with edges e, f, g . The fan is three-dimensional, with all cones containing the line $y_e = y_f = y_g$, so we display a 2-dimensional slice.

Theorem 2.8. *[AHFV25, CD06] The bracket associahedral fan $\text{BrAssocFan}(H)$ is a complete fan in $\mathbb{R}^{E(H)}$. The map $B \mapsto \text{cone}_H(B)$ is an order-preserving bijection between the poset of bracketings $\text{Br}(H)$ and the face lattice $L(\text{BrAssocFan}(H))$ of the bracket associahedral fan.*

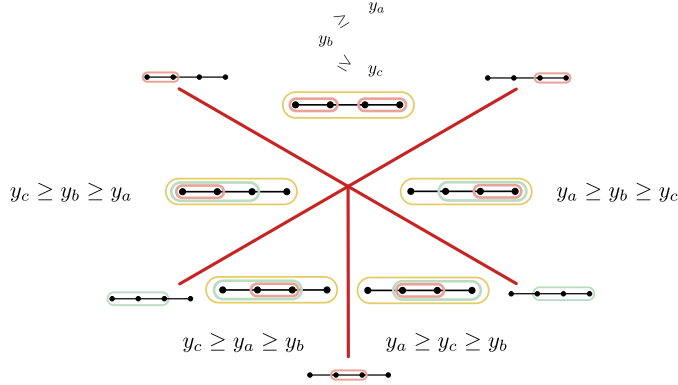


Figure 7: The bracket associahedral fan of a path.

Bracket associahedral fans and graph associahedral fans. The *line graph* $L(H)$ of $H = (V, E)$ is the graph of edge adjacencies of H : it has a vertex for each edge e of H , and the vertices corresponding to edges e, f are connected in $L(H)$ if e and f share a vertex in H . The bracketings of H are in order-preserving bijection with the tubings of $L(H)$ in the sense of [CD06], so

$$\{\text{bracket associahedral fans}\} \subsetneq \{\text{graph associahedral fans}\}.$$

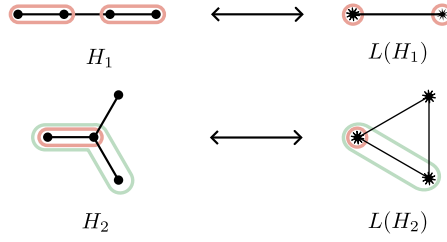


Figure 8: Bracketings of H correspond to tubings of $L(H)$.

The graph associahedral fan of the claw $K_{1,3}$ – studied and depicted in [ARW06, Figure 3] in its manifestation as the Dynkin diagram D_4 – is a three-dimensional fan with 10 rays, corresponding to the 10 tubes of the graph. The reader may verify that this is the only graph on four vertices that has 10 tubes. Since the claw is not a line graph, this is not a bracket associahedral fan. It is then natural to ask:

Which graph associahedral fans are bracket associahedral fans?

2.3.3 The cosmoedral fan in terms of bracketed trees.

A *bracketed tree* (T, B) consists of a plane tree T and a bracketing B of T . Let us define the cone of a bracketed tree (T, B) with $n + 1$ leaves to be

$$\text{cone}(T, B) = \{x \in \mathbb{R}^n : x_i - x_j \geq x_k - x_l \geq 0 \text{ when } \text{bracket}_B(ij) \supseteq \text{bracket}_B(kl)\}.$$

Note that the inequalities of $\text{cone}(T, B)$ correspond to the order relations of the tree poset $\text{tree}(B)$, as illustrated in Figure 9.

The goal of this section is to show that the cones of the bracketed trees on $n + 1$ leaves are precisely the cones of the cosmoedral fan CosmoFan_{n-1} . We begin with a technical lemma.

Lemma 2.9. *For any bracketed tree (T, B) we have*

$$\text{cone}(T, B) \subseteq \text{cone}(T), \quad \dim \text{cone}(T, B) = \dim \text{cone}(T) = |V(T)_{\text{int}}| + 1.$$

Proof. The inclusion and the formula for $\dim \text{cone}(T)$ are clear from the definitions. To prove that $\text{cone}(T, B)$ has the same dimension, we first claim that for $x \in \text{cone}(T, B)$ we have that $x_i = x_j$ if i and j label the same vertex of T . To see this, note that for an offspring k we have $\text{bracket}_B(ik) = \text{bracket}_B(jk)$ so $x_i - x_k = x_j - x_k$ (or analogously for the parent k , if they do not have offspring). Setting these equalities aside, the remaining inequalities of $\text{cone}(T, B)$ are given by the acyclic poset $\text{tree}(B)$, so they imply no further equalities. \square

The bijection with Matryoshkas.

For a Matryoshka M , let $\mathcal{S} = M_{\min}$ be the set of inclusion-minimal polygons in M , which form the underlying subdivision of M .

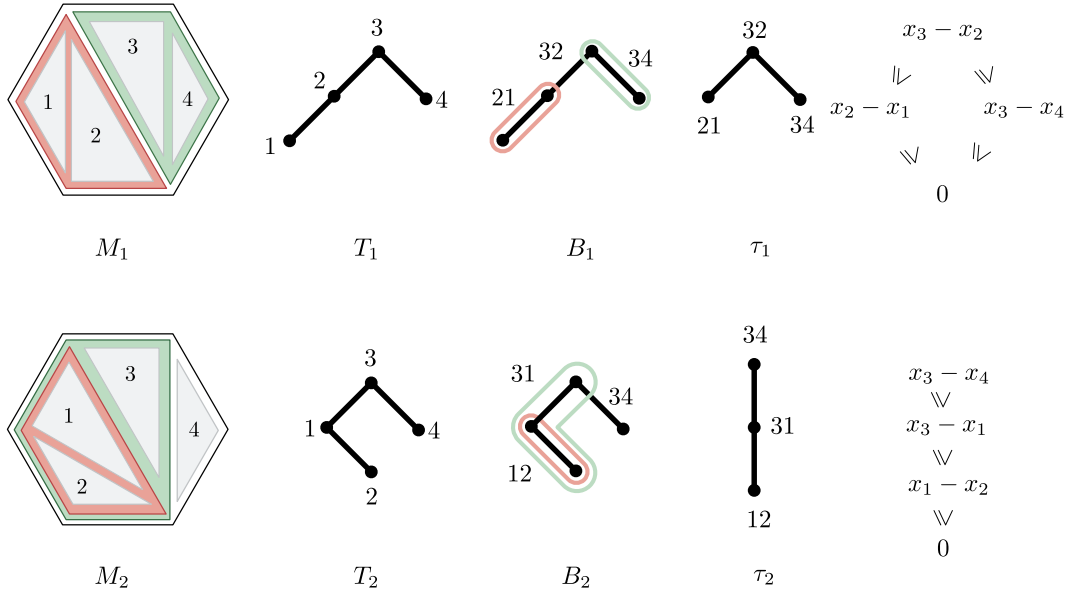


Figure 9: Two Matryoshkas, their corresponding bracketed trees, and their cones.

1. The *rooted planar tree* T_M of M is the dual graph of \mathcal{S} , rooted at the top polygon containing edge e .
2. Each non-minimal polygon P in M is a disjoint union of minimal polygons in M_{\min} , whose dual graph is a bracket of the tree T_M , denoted $\text{bracket}(P)$. The *bracketing* B_M of M is $B_M = \{\text{bracket}(P) : P \in M_{\text{not min}}\}$.

Lemma 2.10. *The map $M \mapsto (T_M, B_M)$ is a bijection between the Matryoshkas on an $(n + 2)$ -gon and the bracketed trees BrTree_{n-1} with $n + 1$ leaves.*

Proof. The Matryoshka condition on M implies that B_M is indeed a bracketing of T_M . Conversely, for any pair (T, B) , the plane tree T is dual to a subdivision \mathcal{S} of the $(n + 2)$ -gon, and each bracket in B is dual to a polygon P that is the union of polygons in \mathcal{S} . The bracketing condition on B implies that these polygons form a Matryoshka. \square

Lemma 2.11. *Under the bijection of Lemma 2.10, the cone of the Matryoshka M equals the cone of the bracketed tree (T_M, B_M) :*

$$\text{cone}(M) = \text{cone}(T_M, B_M).$$

Proof. This follows readily from the definitions. \square

Corollary 2.12. *We have*

$$\text{CosmoFan}_{n-1} = \{\text{cone}(T, B) : (T, B) \text{ is a bracketed tree with } n + 1 \text{ leaves}\}.$$

The poset of bracketed trees. Consider a graph G and a bracketing B . If β is a bracket of B , the *deletion* is the bracketing $B - \beta$ is a bracketing of G . If μ is a minimal bracket of B , the *contraction* is the bracketing $B/\mu := \{\beta - \mu : \beta \in B, \beta \neq \mu\}$ of the tree T/μ where edge μ is contracted – so the edge is deleted and its vertices are identified; see Figure 10. If E is a downset in (the tree pre-poset of) B , we can define B/E by successively contracting the brackets in E in a non-decreasing order.

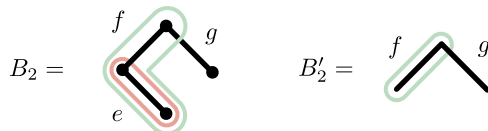


Figure 10: A contraction of a bracketing.

The *poset of bracketed plane trees* with $n + 1$ leaves, denoted BrTree_{n-1} , is defined by deletion and contraction as follows. Locally, we have a cover relation $(T, B) \triangleleft (T', B')$ if either:

- (i) $T = T'$ and $B = B' - \beta'$ for a bracket β' of B' , or
- (ii) $T = T'/\mu'$ and $B = B'/\mu'$ for a minimal bracket μ' of B' .

Globally, we have a poset relation $(T, B) \leq (T', B')$ if $T = T'/E$ for a set of edges E that is a downset of the tree poset of B' , and $B \subseteq B'/E$. We augment this poset with a maximum element $\hat{1}$.

Proposition 2.13. *The bijection $M \mapsto (T_M, B_M)$ of Lemma 2.10, is a poset isomorphism between the lattice of Matryoshkas Mat_{n+2} and the poset of bracketed trees BrTree_{n-1} .*

Proof. In a cover relation $M \triangleleft M'$ of Matryoshkas, M is obtained from M' by either:
(i) removing a non-minimal polygon P , which corresponds to leaving $T_M = T_{M'}$ fixed and making $B_M = B_{M'} - \text{bracket}(P)$, or
(ii) removing the minimal polygons P_1, \dots, P_k that subdivided a next-to-minimal polygon P in M' , so P becomes minimal in M ; this corresponds to contracting the minimal bracket (P) to get $(T_M, B_M) = (T_{M'}/\text{bracket}(P), B_{M'}/\text{bracket}(P))$.

This proves the poset isomorphism. \square

3 Proof of correctness of the cosmoedral fan

3.1 Positive bracket associahedral fans

The *positive bracket associahedral fan* of a graph H is the intersection

$$\text{BrAssocFan}_{\geq 0}(H) = \text{BrAssocFan}(H) \cap \mathbb{R}_{\geq 0}^{E(H)}$$

of the bracket associahedral fan with the positive orthant, with the inherited polyhedral structure.

$F = E(H) - E$, then

$$\text{cone}_H(B) \cap \mathbb{R}_{\geq 0}^E = \text{cone}_H(B/\overline{F})_{\geq 0}. \quad (2)$$

Every pre-bracketing of H arises in this way, proving the description of the cones of $\text{BrAssoc}_{\geq 0}(H)$.

Now, for a pre-bracketing B of H , a facet of $\text{cone}_H(B)_{\geq 0}$ is obtained by either (i) setting $y_e = y_f$ for $e \succ f$ in $\text{tree}(B)$, which has the effect of deleting the bracket of f , or (ii) setting $y_f = 0$ for a minimal f in $\text{tree}(B)$, which has the effect of contracting the bracket of f . The poset isomorphism follows. \square

3.2 The cosmohedral fan and positive bracket associahedral fans

Now we describe the cosmohedral fan in terms of positive bracket associahedral fans.

Lemma 3.3. *Let T be a plane rooted tree with $n + 1$ leaves, whose internal vertices are canonically labelled by $[n]$. Label each edge in the order ij where i is closer to the root than j . We have inverse maps:*

$$\begin{aligned} f_T : \mathbb{R}^{E(T)} &\xrightarrow{\cong} \mathbb{R}^n / \mathbb{R}\mathbf{1} & f_T^{-1} : \mathbb{R}^n / \mathbb{R}\mathbf{1} &\xrightarrow{\cong} \mathbb{R}^{E(T)} \\ e_{ij} &\longmapsto \sum_{k \in (T-ij)_i} e_k + \mathbb{R} & (x_v)_{v \in V(T)} + \mathbb{R}\mathbf{1} &\longmapsto (x_i - x_j)_{ij \in E(T)}. \end{aligned}$$

where $(T-ij)_i$ and $(T-ij)_j$ are the connected components of the forest $T-ij$ containing i and j respectively. Furthermore,

1. the map $f_T : \mathbb{R}^{E(T)} \xrightarrow{\cong} \mathbb{R}^n / \mathbb{R}\mathbf{1}$ is an isomorphism of vector spaces,
2. the restriction $f_T : \mathbb{R}_{\geq 0}^{E(T)} \xrightarrow{\cong} \text{cone}(T)$ is an isomorphism of cones, and
3. the restriction $f_T : \text{cone}_T(B)_{\geq 0} \xrightarrow{\cong} \text{cone}(T, B)$ is an isomorphism of cones for any bracketing B of tree T .
4. the restriction $f_T : \text{cone}_T(B)_{\geq 0} \xrightarrow{\cong} \text{cone}(T(B), B)$ is an isomorphism of cones for any pre-bracketing B of tree T , where $T(B) \leq T$ is the tree supporting B .

Proof. First we check that these maps are indeed inverse to each other. For each basis vector e_{ij} corresponding to edge ij of T , the vector $f_T(e_{ij})$ has coordinates equal to 1 on component $(T-ij)_i$ and 0 on component $(T-ij)_j$. Therefore $f_T^{-1}(f_T(e_{ij}))$ is only non-zero on the edge ij – whose vertices lie in different components – where it equals 1. Thus $f_T^{-1}(f_T(e_{ij})) = e_{ij}$ indeed. Since both spaces are $(n-1)$ -dimensional, this means f_T is an isomorphism and f_T^{-1} is its inverse, proving 1.

To show 2. notice that $y = f_T^{-1}(x)$ is nonnegative if and only if x satisfies the inequalities of $\text{cone}(T)$. For 3., observe that y satisfies the additional inequalities $y_{ij} \geq y_{kl} \geq 0$ of $\text{cone}_T(B)_{\geq 0}$ if and only if x satisfies the inequalities $x_i - x_j \geq x_k - x_l \geq 0$ of

$\text{cone}(T, B)$ for $ij \geq_B kl$. For 4. notice that y satisfies $y_{ij} = 0$ for each contracted edge ij if and only if x satisfies $x_i = x_j$ when i and j label the same vertex in $T(B)$; these are precisely the equalities that hold in $\text{cone}(T(B), B)$, as we explained in the proof of Lemma 2.9. \square

Example 3.4. Consider the Matryoshka M_1 and its corresponding tree T_1 with edges $e = 21, f = 32, g = 34$, and bracketing $B_1 = \{e, efg, g\}$, as shown in Figure 9. Then

$$\text{cone}_{T_1}(B_1)_{\geq 0} = \{y \in \mathbb{R}^{efg} : y_f \geq \{y_e, y_g\} \geq 0\} \subset \mathbb{R}^{E(T)},$$

and its image under the map f_{T_1} is

$$\text{cone}(T_1, B_1) = \{x \in \mathbb{R}^4 / \mathbb{R}\mathbf{1} : x_3 - x_2 \geq \{x_2 - x_1, x_3 - x_4\} \geq 0\} \subset \mathbb{R}^n / \mathbb{R}\mathbf{1}$$

as illustrated in Figure 12.

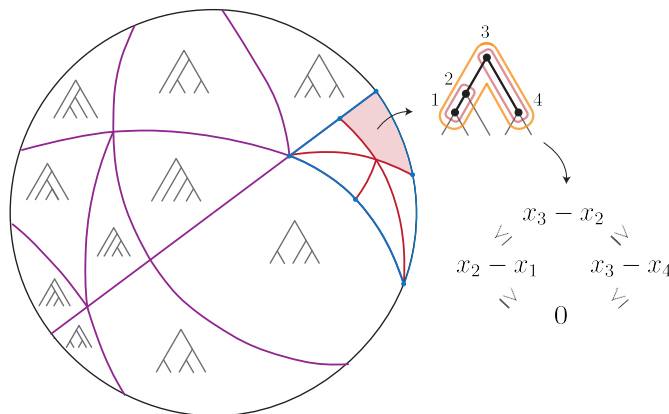


Figure 12: The linear map f_{T_1} embeds the four-sided $\text{cone}_{T_1}(B_1)_{\geq 0}$ of $\text{BrAssocFan}_{\geq 0}(T_1)$ in Figure 11 into the $\text{cone}(T_1)$ of Assoc_{n-1} in Figure 6. The result is the cone $f_{T_1}(\text{cone}_{T_1}(B_1)_{\geq 0}) = \text{cone}(T_1, B_1)$ of Cosmo_{n-1} .

For the moment, we only know CosmoFan_{n-1} to be the set of Matryoshkal cones. Our next goal is to show that it is indeed a polyhedral fan, with the correct combinatorial structure. We begin with a simple technical lemma.

Corollary 3.5. We have

$$\text{CosmoFan}_{n-1} = \bigcup_T f_T(\text{BrAssocFan}_{\geq 0}(T))$$

where the union is over all binary trees T with $n + 1$ leaves.

Proof. Any cone of CosmoFan_{n-1} is of the form $\text{cone}(M) = \text{cone}(T, B)$ for a bracketed tree (T, B) . If we choose any binary $T^+ \geq T$, then B is a pre-bracketing of T^+ , and Lemma 3.3.4 shows that this cone is in $f_{T^+}(\text{BrAssocFan}_{\geq 0}(T^+))$. \square

Notice in particular that the union above is not disjoint.

3.3 The cosmohedral fan is a complete fan

Theorem 3.6. *The cosmohedral fan CosmoFan_{n-1} is a complete fan in $\mathbb{R}^n/\mathbb{R}\mathbf{1}$. The map $M \mapsto \text{cone}(M)$ is an order-preserving bijection between the Matryoshkas of an $(n+2)$ -gon and the faces of CosmoFan_{n-1} .*

Proof. To prove that the cones intersect properly, let us first analyze how a Matryoshkal cone $\text{cone}(T_1, B_1)$, which sits in an associahedral cone $\text{cone}(T_1)$ of the same dimension, intersects a smaller associahedral cone $\text{cone}(T) \subseteq \text{cone}(T_1)$. Consider any maximal $T^+ \geq T_1$. Then B_1 is a pre-bracketing of T^+ and we have

$$\begin{aligned} \text{cone}(T_1, B_1) \cap \text{cone}(T) &= f_{T^+}(\text{cone}_{T^+}(B_1)_{\geq 0}) \cap f_{T^+}(\mathbb{R}_{\geq 0}^{E(T)}) \\ &= f_{T^+}(\text{cone}_{T^+}(B_1)_{\geq 0} \cap \mathbb{R}_{\geq 0}^{E(T)}) \\ &= f_{T^+}(\text{cone}_{T^+}(B_1/\overline{F})_{\geq 0}) \quad \text{for } F = E(T^+) - E(T), \text{ by (2)} \\ &= \text{cone}(T'_1, B'_1) \quad \text{for } B'_1 = B_1/\overline{F} \end{aligned}$$

where $T'_1 = T^+/\overline{F} = T/(\overline{F} - F) \leq T$ is the tree supporting B'_1 .

For example, in the top row of Figure 13 we compute $\text{cone}(M_1) \cap \text{cone}(T)$ for the Matryoshka $M_1 = (T_1, B_1)$ of Figure 9 and the tree T of the shown subdivision. Here $T^+ = T_1$, $F = E(T^+) - E(T) = \{21\}$, and the downset in τ_1 is $\overline{F} = F = \{21\}$. Thus we obtain (T'_1, B'_1) by contracting edge 21 from (T_1, B_1) . The resulting inequalities are $x_3 - x_{21} \geq x_3 - x_4 \geq 0$, where x_{21} denotes the common value of $x_2 = x_1$. In the second row we compute $\text{cone}(M_2) \cap \text{cone}(T)$ similarly.

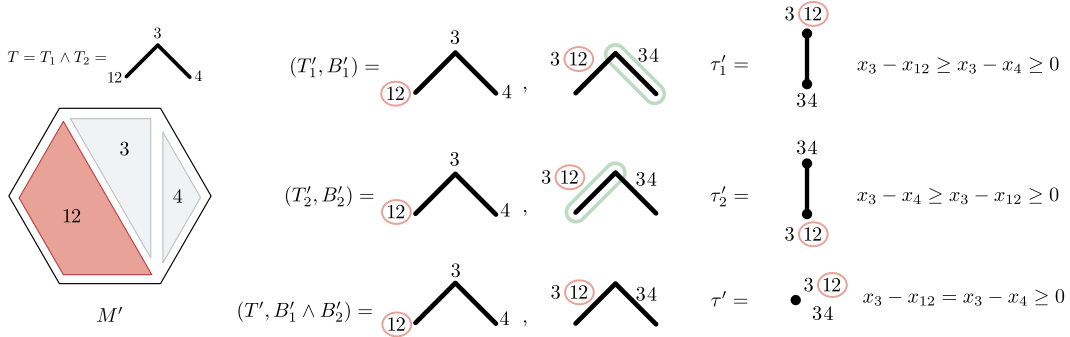


Figure 13: The intersection of the two Matryoshkal cones M_1 and M_2 described in Figure 9. We begin by computing $T = T_1 \wedge T_2$ and use it to find T'_1, T'_2 and the inherited bracketings B'_1 and B'_2 . From there we find $B' = B'_1 \wedge B'_2$. Then M' is the Matryoshka of T' and B' .

Now consider two Matryoshkas $M_1 = (T_1, B_1)$ and $M_2 = (T_2, B_2)$. Their cones

intersect inside $\text{cone}(T_1) \cap \text{cone}(T_2) = \text{cone}(T)$ where $T = T_1 \wedge T_2$ in Tree_{n-1} . Then

$$\begin{aligned}
\text{cone}(M_1) \cap \text{cone}(M_2) &= (\text{cone}(T_1, B_1) \cap \text{cone}(T)) \cap (\text{cone}(T_2, B_2) \cap \text{cone}(T)) \\
&= \text{cone}(T'_1, B'_1) \cap \text{cone}(T'_2, B'_2) \text{ where } T'_1, T'_2 \leq T \\
&= f_{T^+}(\text{cone}_{T^+}(B'_1)_{\geq 0} \cap \text{cone}_{T^+}(B'_2)_{\geq 0}) \text{ for any maximal } T^+ \geq T \\
&= f_{T^+}(\text{cone}_{T^+}(B')_{\geq 0}) \text{ for } B' = B'_1 \wedge B'_2, \text{ by Proposition 3.2} \\
&= \text{cone}(M') \text{ for the Matryoshka } M' \text{ of } (T', B')
\end{aligned}$$

where $T' \leq T^+$ is the tree supporting $B' = B'_1 \wedge B'_2$.

In the third row of Figure 13 we compute $\text{cone}(M_1) \cap \text{cone}(M_2)$ for the Matryoshkas of Figure 9, noticing that the tree we had considered is $T = T_1 \wedge T_2$. Considering B'_1 and B'_2 as pre-bracketings of any maximal tree $T^+ \geq T$ we compute their meet $B' = B'_1 \wedge B'_2$ and obtain from it the tree T' and the inequalities of the intersection, $x_3 - x_{12} = x_3 - x_4 \geq 0$; that is, the ray $x_3 \geq x_{124}$ in \mathbb{R}^4/\mathbb{R} .

To prove that the fan is complete, we begin with the fact that the maximal tree cones $\text{cone}(T)$ of the associahedral fan cover $\mathbb{R}^n/\mathbb{R}\mathbf{1}$. Now, for each T , the cones $\text{cone}_T(B)_{\geq 0}$ of the pre-bracketing fan cover the positive orthant $\mathbb{R}_{\geq 0}^{E(T)}$, so their images $f_T(\text{cone}_T(B)_{\geq 0}) = \text{cone}(T(B), B)$ cover $f_T(\mathbb{R}_{\geq 0}^{E(T)}) = \text{cone}(T)$.

Finally let us prove the poset isomorphism, keeping in mind that we already showed that $\text{Mat}_{n+2} \cong \text{BrTree}_{n+1}$. Clearly $M_1 \subseteq M_2$ implies $\text{cone}(M_1) \subseteq \text{cone}(M_2)$. Assume that $\text{cone}(M_1) = \text{cone}(T_1, B_1) \subseteq \text{cone}(T_2, B_2) = \text{cone}(M_2)$. Let T be a maximal tree such that $\text{cone}(T)$ contains these cones; B_1 and B_2 are pre-bracketings of T . Then $f_T(\text{cone}_T(B_1)_{\geq 0}) = \text{cone}(T_1, B_1) = \text{cone}(T_2, B_2) \subseteq f_T(\text{cone}_T(B_2)_{\geq 0})$ by Lemma 3.3 so $\text{cone}_T(B_1)_{\geq 0} \subseteq \text{cone}_T(B_2)_{\geq 0}$ which means that $B_1 \leq B_2$ in $\text{PreBr}(T)$ by Proposition 3.2. Therefore $(T_1, B_1) \subseteq (T_2, B_2)$ in BrTree_{n-1} , which means that $M_1 \leq M_2$ in Mat_{n-1} by Proposition 2.13. \square

4 The cosmohedron

In this section we define the cosmohedron $\text{Cosmo}_{n-1}(\mathbf{a}, \mathbf{b})$ and prove that its normal fan is the cosmohedral fan CosmoFan_{n-1} .

To set conventions, let e_1, \dots, e_n be the standard basis of \mathbb{R}^n . Let us write $i \ll j$, when integers $i < j$ are not consecutive, that is, $|j - i| > 1$. We note that the set of pairs (i, j) with $0 \leq i \ll j \leq n+1$ is in bijection with the set $\text{diag}^+(\diamond_{n+2}) = \text{diag}(\diamond_{n+2}) \cup \{e\}$ of diagonals augmented with the base edge e .

4.1 Preliminaries: Associahedra and bracket associahedra

Loday's associahedron. Given a sequence of positive numbers $\mathbf{a} = (a_{ij})_{0 \leq i < j \leq n+1}$, the *Loday associahedron* is

$$\begin{aligned} \text{Loday}_{n-1}(\mathbf{a}) &= \sum_{0 \leq i < j \leq n+1} a_{ij} \Delta_{(i,j)} & (3) \\ &= \{x \in \mathbb{R}^n : \sum_{1 \leq i \leq n} x_i = \sum_{0 \leq i < j \leq n+1} a_{ij}, \\ &\quad \sum_{r \leq i \leq s} x_i \geq \sum_{r-1 \leq i < j \leq s+1} a_{ij} \text{ for } 1 \leq r \leq s \leq n.\} & (4) \end{aligned}$$

where $\Delta_{(i,j)} = \Delta_{[i+1,j-1]} = \text{conv}(e_{i+1}, \dots, e_{j-1})$ is a face of the standard simplex in \mathbb{R}^n , and $P + Q = \{p + q : p \in P, q \in Q\}$ denotes Minkowski sum of polytopes.¹

Theorem 4.1. [AA23, Pos09] *The normal fan of the associahedron $\text{Loday}_{n-1}(\mathbf{a})$ is the associahedral fan AssocFan_{n-1} .*

Let \mathcal{S} be a triangulation of \diamond_{n+2} corresponding to the binary tree T . It will be useful to express the vertex a_T of Loday's associahedron in terms of \mathcal{S} . To do so, say that diagonal ij *transversally crosses* the triangle labeled v with vertices $u < v < w$ if it intersects edges uv and vw , not at v . This is equivalent to $u \leq i < v < j \leq w$. When this is the case we write $ij \rightarrow v$. We invite the reader to check that each diagonal of the polygon crosses exactly one triangle of the triangulation \mathcal{S} transversally.

Lemma 4.2. [Lod04, Pos09] *The vertex $a_{\mathcal{S}}$ of $\text{Loday}_{n-1}(\mathbf{a})$ corresponding to a triangulation \mathcal{S} of \diamond_{n+2} has coordinates*

$$(a_{\mathcal{S}})_v = \sum_{ij \rightarrow v \text{ in } \mathcal{S}} a_{ij}$$

summing over the extended diagonals $ij \in \text{diag}^+(\diamond_{n+2})$ that cross triangle v transversally.

Proof. The vertex a_T of $\text{Loday}_{n-1}(\mathbf{a})$ corresponding to a binary tree T has coordinates

$$(a_T)_v = \sum_{i \vee j = v \text{ in } T} a_{i-1, j+1} \quad \text{for } v \in [n]$$

where $i \vee j$ is the lowest common ancestor of i and j ; this is implicit in [Lod04, Pos09] see for example [AA23, Figure 6]. The second statement then follows readily from the bijection. \square

Devadoss's bracket associahedron. For a graph H , let $\mathbf{b} : \text{Brackets}(H) \rightarrow \mathbb{Z}$ be any function that is *concave* in the sense that $b(\beta_1) + b(\beta_2) > b(\beta_1 \cap \beta_2) + b(\beta_1 \cup \beta_2)$ for any intersecting brackets β_1 and β_2 . For example, $b(\beta)$ can be any concave function of the number of edges $|\beta| = k$; that is, $b(k) > \frac{1}{2}(b(k-1) + b(k+1))$.

¹It is more customary to choose $\alpha = (\alpha_{ij})_{1 \leq i \leq j \leq n}$ and write $\sum_{i \leq j} \alpha_{ij} \Delta_{[i,j]}$ for Loday's associahedron; we use $a_{i-1, j+1} = \alpha_{ij}$ instead.

Definition 4.3. *The Devadoss bracket associahedron of a graph H is*

$$\begin{aligned} \text{BrAssoc}_H(\mathbf{b}) = \{y \in \mathbb{R}^{E(H)} : \sum_{e \in E(H)} y_e = b(H), \\ \sum_{e \in \beta} y_e \leq b(\beta) \text{ for all brackets } \beta \text{ of } H\}. \end{aligned} \quad (5)$$

Theorem 4.4. *[Dev09, PPP23] The normal fan of the bracket associahedron $\text{BrAssoc}_H \mathbf{b}$ is the bracket associahedral fan of H .*

We remark that Devadoss [Dev09] works with the sequence $b(\beta) = -3^{|\beta|-2}$ – see Lemma 5.2 – but in fact Theorem 4.4 holds precisely for concave sequences. To see this, we use Padrol, Pilaud, and Poullot’s description [PPP23, Corollary 2.12] of the deformation cone of graph associahedral fans, and in particular bracket associahedral fans. This cone is given by the equality $b(G) = 0$ and inequalities $b(\beta - e) + b(\beta - e') < b(\beta) + b(\beta - e - e')$ for any bracket β and any non-disconnecting edges e, e' of β . This is equivalent to the concavity condition above.²

Lemma 4.5. *Let $\mathbf{b} : \text{Brackets}(H) \rightarrow \mathbb{R}$ be concave. The vertex b_B of $\text{BrAssoc}_H(\mathbf{b})$ associated to a maximal bracketing B of H is given by*

$$(b_B)_e = b(\text{bracket}(e)) - \sum_{f \triangleleft e} b(\text{bracket}(f)) \quad \text{for } e \in E(H), \quad (6)$$

summing over the edges f that e covers in $\text{tree}(B)$.

In a maximal bracketing each bracket contains one or two maximal sub-brackets, so the expression above has at most three terms.

Proof. For each maximal bracketing B , we can compute the entries of vertex b_B recursively by $(b_B)_e = b(e)$ for any edge e that is a bracket, and $\sum_{e \in \beta} (b_B)_e = b(\beta)$ for every bracket β of B . In the tree of B , this says $b(\text{bracket}(e)) = \sum_{f \triangleleft e} (b_B)_e$. Adding a minimum element $\widehat{0}$ with $b(\widehat{0}) = b_B(\widehat{0}) = 0$, and applying Möbius inversion in the poset $\text{tree}(B) \cup \widehat{0}$, we get $(b_B)_e = \sum_{f \triangleleft e} \mu(f, e) b(\text{bracket}(f))$. Since the interval $[f, e]$ is a path, $\mu(f, e)$ equals 1 if $f = e$, it equals -1 if $f \triangleleft e$, and it is 0 otherwise; so (6) does indeed describe Devadoss’s bracket associahedron. \square

Slightly differently, we will work with the *extended set of brackets* $\text{Brackets}^+(H) = \text{Brackets}(H) \cup V(H)$, including also a *small bracket* around each individual vertex. Consider any concave function $\mathbf{b} : \text{Brackets}^+(H) \rightarrow \mathbb{Z}$; in the concavity condition, if $\beta_1 \cap \beta_2$ is an edge-less graph on vertices v_1, \dots, v_k , we set $b(\beta_1 \cap \beta_2) = b(v_1) + \dots + b(v_k)$. The function \mathbf{b} induces a concave function $\mathbf{b}^- : \text{Brackets}(H) \rightarrow \mathbb{Z}$ defined by

$$b^-(\beta) = b(\beta) - \sum_{v \in V(\beta)} b(v), \quad (7)$$

²Padrol, Pilaud, and Poullot require $b(G) = 0$, but we can remove this condition by a suitable translation. For general graphs some care is required when $\beta - e - e'$ is disconnected, but this does not happen for trees.

whose bracket associahedron we now describe. A bracketing B of H now gives an *extended tree*⁺(B) that also includes the small brackets.

Lemma 4.6. *Let $\mathbf{b} : \text{Brackets}^+(H) \rightarrow \mathbb{R}$ be concave and $\mathbf{b}^- : \text{Brackets}(H) \rightarrow \mathbb{R}$ be given by (7). The vertex b_B of $\text{BrAssoc}_H(\mathbf{b}^-)$ associated to a maximal bracketing B of H is given by*

$$(b_B)_e = b(\text{bracket}(e)) - \sum_{f < e} b(\text{bracket}(f)) \quad \text{for } e \in E(H). \quad (8)$$

summing over the edges f that e covers in the extended tree⁺(B).

Proof. This follows by applying Lemma 4.5 to \mathbf{b}^- . □

4.2 The cosmohedron

In this section we construct a *cosmohedron*: a polytope $\text{Cosmo}_{n-1}(\mathbf{a}, \mathbf{b})$ whose normal fan is the cosmohedral fan CosmoFan_{n-1} . We first give two vertex descriptions of $\text{Cosmo}_{n-1}(\mathbf{a}, \mathbf{b})$ and use them to prove that it has the correct normal fan. We then use the normal fan to compute the inequality description of $\text{Cosmo}_{n-1}(\mathbf{a}, \mathbf{b})$. Finally, we use those inequalities to prove that $\text{Cosmo}_{n-1}(\mathbf{a}, \mathbf{b})$ is linearly isomorphic to the cosmohedron proposed by Arkani-Hamed, Figueiredo, and Vazão [AHFV25]. This gives a proof of the correctness of their construction.

Recall that $\text{diag}^+(\diamond_{n+2}) = \text{diag}(\diamond_{n+2}) \cup e$ is the set of diagonals of the $(n+2)$ -gon augmented with the base e . We say a function $\mathbf{b} = (b(P) : P \text{ is a subpolygon of } \diamond_{n+2}) \in \mathbb{R}^{\binom{n+2}{\geq 3}}$ is *concave* if $b(P_1) + b(P_2) > b(P_1 \cap P_2) + b(P_1 \cup P_2)$ whenever the intersection and union are also subpolygons. An example is a concave function of the number of vertices of the polygon.

Throughout this section we fix:

- an arbitrary positive vector $\mathbf{a} = (a_{ij} : ij \in \text{diag}^+(\diamond_{n+2})) \in \mathbb{R}^{\text{diag}^+(\diamond_{n+2})} \cong \mathbb{R}^{\binom{n+1}{2}}$,
- a concave vector $\mathbf{b} = (b(P) : P \text{ is a subpolygon of } \diamond_{n+2}) \in \mathbb{R}^{\binom{n+2}{\geq 3}}$,
- a small constant $0 < \varepsilon < \frac{m}{24M}$ where $m = \min a_{ij}$ and $M = \max |b(P)|$.

The cosmohedron $\text{Cosmo}_{n-1}(\mathbf{a}, \mathbf{b})$ is constructed from those choices.

4.2.1 Vertex description

For a rooted plane tree T let

$$\begin{aligned} f^T : \mathbb{R}^{E(T)} &\longrightarrow \mathbb{R}_0^{V(T)} \\ e_{ij} &\longmapsto e_i - e_j \end{aligned}$$

be the isomorphism between $\mathbb{R}^{E(T)}$ and $\mathbb{R}_0^{V(T)} := \{x \in \mathbb{R}^{V(T)} : \sum_v x_v = 0\}$ dual to the isomorphism f_T of Lemma 3.3. Also, our chosen vector $\mathbf{b} \in \mathbb{R}^{\binom{n+2}{\geq 3}}$ induces a map $\mathbf{b} : \text{Brackets}^+(T) \rightarrow \mathbb{R}$, since the extended brackets of T correspond to the polygons in the triangulation of M . In turn this induces a map $\mathbf{b}^- : \text{Brackets}(T) \rightarrow \mathbb{R}$ by (7).

Definition 4.7. For a maximal Matryoshka M with bracketed tree (T, B) we define

$$c_M := a_T + \varepsilon f^T(b_B)$$

where a_T is the vertex of $\text{Loday}_{n-1}(\mathbf{a})$ corresponding to the plane tree T and b_B is the vertex of $\text{BrAssoc}_T(\mathbf{b}^-)$ corresponding to its bracketing B . The cosmohedron is

$$\text{Cosmo}_{n-1}(\mathbf{a}, \mathbf{b}) := \text{conv}(c_M : M \text{ is a maximal Matryoshka on the } (n+2)\text{-gon}) \subset \mathbb{R}^n.$$

Before we prove the correctness of this construction, let us give an explicit formula for the vertex c_M directly in terms of the Matryoshka M .

Proposition 4.8. Let M be a maximal Matryoshka on \diamond_{n+2} . For each $v \in [n]$ define

$$(a_M)_v = \sum_{ij \rightarrow v} a_{ij} = \sum_{u \leq i < v} \sum_{v < j \leq w} a_{ij}$$

summing over the diagonals ij that cross triangle v (with vertices uvw) of the triangulation of M transversally.

For each polygon of the Matryoshka M that is subdivided into maximal subpolygons Q and R in M , let

$$b_M(P) = b(P) - b(Q) - b(R).$$

For each $v \in [n]$, where triangle v has edges u_v (opposite to vertex v), l_v , and r_v , define

$$(b_M)_v = b_M(P(l_v)) + b_M(P(r_v)) - b_M(P(u_v))$$

where $P(e)$ is the minimal polygon in M containing diagonal e ; when e is an edge of \diamond_{n+2} , we set $b_M(P(e)) = 0$. Then the vertex c_M of a Matryoshka M has coordinates

$$(c_M)_v = (a_M)_v + \varepsilon (b_M)_v \quad \text{for } v \in [n].$$

Proof. This follows readily from Lemmas 4.2 and 4.5 and Definition 4.7. \square

Remark 4.9. Our preferred choice of \mathbf{a} and \mathbf{b} will be

$$a_{ij} = 1 \text{ for all } i, j, \quad b(P) = \begin{cases} 0 & \text{if } |P| \leq 4 \\ 3^{|P|-5} & \text{if } |P| \geq 5. \end{cases}$$

This choice simplifies computations: we have

$$(a_M)_v = |l_v| \cdot |r_v|$$

where $|f|$ is the number of edges of \diamond_{n+2} that are on the side of f opposite to e . Also $b_M(P) = 0$ for all triangles and quadrilaterals P .

Example 4.10. Let us compute the vertex c_{M_2} for the Matryoshka M_2 of Figure 4. We have $a_{M_2} = (1 \cdot 2, 1 \cdot 1, 3 \cdot 2, 1 \cdot 1) = (2, 1, 6, 1)$. The non-zero coordinates of b_{M_2} are $b_{M_2}(\text{hexagon}) = 3 - 1 - 0 = 2$ and $b_{M_2}(\text{pentagon}) = 1 - 0 - 0 = 1$. Therefore $b_{M_2} = (0 + 0 - 1, 0 + 0 - 0, 1 + 2 - 0, 0 + 0 - 2) = (-1, 0, 3, -2)$ and

$$c_{M_2} = (2, 1, 6, 1) + \varepsilon(-1, 0, 3, 2).$$

4.2.2 The cosmohedron: proof of correctness

Theorem 4.11. *The normal fan of the cosmohedron $\text{Cosmo}_{n-1}(\mathbf{a}, \mathbf{b})$ is the cosmohedral fan CosmoFan_{n-1} .*

Proof. Let $\mathcal{N}_P(v) := \{w \in V^* : w(v) \geq w(p) \text{ for all } p \in P\}$ denote the normal cone of vertex v in polytope $P \subset V$. Since the cosmohedron lives on a hyperplane with constant coordinate sum, its dual space is $\mathbb{R}^n/\mathbb{R}\mathbf{1}$. We need to prove that

$$\mathcal{N}_{\text{Cosmo}_{n-1}(\mathbf{a}, \mathbf{b})}(c_M) = \text{cone}(M)$$

for all maximal Matryoshkas M . Since both the left-hand sides and the right-hand sides subdivide $\mathbb{R}^n/\mathbb{R}\mathbf{1}$ with no interior overlap as M ranges over the maximal Matryoshkas, it suffices to prove the inclusion \subseteq for all M .

Let M be a maximal Matryoshka with bracketed tree (T, B) , and take $w \in \mathcal{N}_{\text{Cosmo}_{n-1}}(c_M)$. To prove that $w \in \text{cone}(M)$, we proceed in two steps.

Step 1. Consider a covering relation $\diamond_M(ij^\perp) \succ \diamond_M(kl^\perp)$ in M . We need to show that w satisfies the corresponding inequality $w_i - w_j \geq w_k - w_l$ for $\text{cone}(M)$.

Edge kl^\perp creates a subdivision $\diamond_M(kl^\perp) = P_1 \cup P_2$ and edge ij^\perp creates a subdivision $\diamond_M(ij^\perp) = \diamond_M(kl^\perp) \cup P_3$. Consider the Matryoshka M' obtained by removing $\diamond_M(kl^\perp)$ and adding a new polygon around P_2 and P_3 .

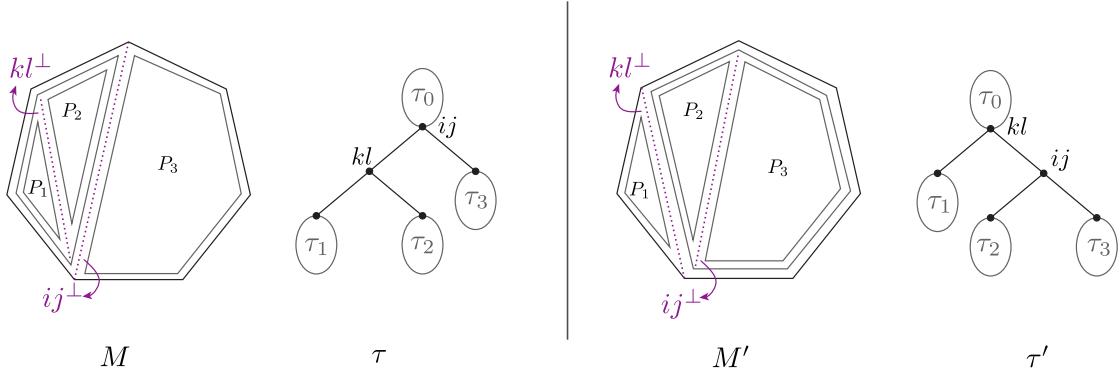


Figure 14: A swap of a non-minimal polygon in a Matryoshka, and the corresponding effect on the polygon containment tree τ .

The bracketed tree of M' is (T, B') for the same tree T and a different bracketing. The containment trees $\tau = \text{tree}(B)$ and $\tau' = \text{tree}(B')$ differ by a simple mutation, as shown in Figure 14. Since Devadoss's bracket associahedron is a generalized permutahedron, its adjacent vertices b_B and $b_{B'}$ satisfy $b_B - b_{B'} = L(e_{ij} - e_{kl})$ for some $L > 0$ [AA23, Pos09]. Now $w \in \mathcal{N}_{\text{Cosmo}_{n-1}}(c_M)$ gives $w(c_M) \geq w(c_{M'})$, so

$$\begin{aligned} w(a_T + \varepsilon f^T(b_B)) &\geq w(a_T + \varepsilon f^T(b_{B'})) \\ w f^T(L(e_{ij} - e_{kl})) &\geq 0 \\ w((e_i - e_j) - (e_k - e_l)) &\geq 0 \end{aligned}$$

as desired.

Step 2. Consider a polygon $\diamond_M(kl^\perp)$ that does not contain a smaller $\diamond_M(k'l'^\perp)$; it must be a quadrilateral split into two triangles k and l . Consider the Matryoshka M' obtained by flipping those two triangles in the quadrilateral.

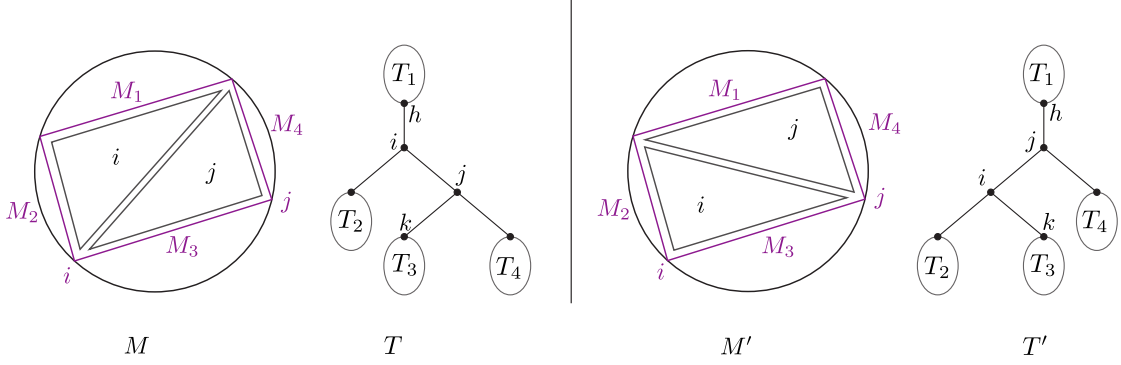


Figure 15: A flip of the triangulation in a Matryoshka, and the corresponding effect on the tree T of the triangulation.

The tree T' of (the triangulation of) M' is obtained from T by a simple edge mutation, as shown in Figure 15. Since Loday's associahedron is a generalized permutahedron, its adjacent vertices a_T and $a_{T'}$ satisfy $a_T - a_{T'} = L(e_i - e_j)$ for some $L > 0$ [AA23, Pos09]. The bracketings B and B' both contain bracket ij , so they contain the same set of polygons, and give isomorphic posets $\tau = \text{tree}(B) \cong \text{tree}(B') = \tau'$. This means that $b_B = b_{B'}$ up to the resulting reordering of the edges $(hi, ij, jk) \mapsto (hj, ji, ik)$. Therefore if we write $H = (b_B)_{hi}$, $I = (b_B)_{ij}$, $J = (b_B)_{jk}$,

$$\begin{aligned} f^T(b_B) - f^{T'}(b_{B'}) &= (H(e_h - e_i) + I(e_i - e_j) + J(e_j - e_k)) \\ &\quad - (H(e_h - e_j) + I(e_j - e_i) + J(e_i - e_k)) \\ &= (-H + 2I - J)(e_i - e_j). \end{aligned}$$

Now since $w \in \mathcal{N}_{\text{Cosmo}_{n-1}}(c_M)$, we have $w(c_M) \geq w(c_{M'})$, so

$$\begin{aligned} w\left((a_T - a_{T'}) + \varepsilon(f^T - f^{T'})(b_B)\right) &\geq 0 \\ w((L + 2\varepsilon(-H + 2I - J))(e_i - e_j)) &\geq 0 \\ w(e_i - e_j) &\geq 0 \end{aligned}$$

as desired. In the last step we are observing that each entry of (b_B) is at most $3M$ because the formula in Lemma 4.5 contains at most three terms, so $|2\varepsilon(-H + 2I - J)| < 24M\varepsilon$, whereas each edge $a_T - a_{T'}$ of $\text{Loday}_{n-1}(\mathbf{a})$ is a sum of edges of its Minkowski summands, so its length L is at least m . \square

Remark 4.12. In Step 2 of the proof above, it is essential that b_B and $b_{B'}$ are equal up to the natural reordering of their coordinates. This is a subtle requirement, because b_B and $b_{B'}$ are vertices of different bracket associahedra BrAssoc_T and $\text{BrAssoc}_{T'}$. The key property of Devadoss's realization of bracket associahedra that makes this work is that the coordinates of a vertex b_B only depend on the poset $\tau = \text{tree}(B)$ of the bracketing, and in the above situation we have $\tau = \text{tree}(B) = \text{tree}(B') = \tau'$. We note that Postnikov's realization [Pos09] of bracket associahedra does **not** have this property, so we cannot use it to realize the cosmohedron in this manner.

4.2.3 Inequality description.

For a chord C and a vertex i or diagonal ij of \diamond_{n+2} , let us write $C \succ i$ if the chord C is strictly above vertex i and $C \succeq ij$ if the chord C is weakly above diagonal ij , with respect to the base edge e at the top.

For a subdivision \mathcal{S} of \diamond_{n+2} (or equivalently a non-crossing set of chords \mathcal{C}), we define some parameters of \mathcal{S} and \mathcal{C} in a few equivalent ways. Each one of the descriptions is sufficient, but it can be useful at times to have all of them available. Let the *depth* of a vertex i or a diagonal ij of \diamond_{n+2} be

$$\begin{aligned} d_{\mathcal{S}}(i) &= \# \text{ of chords } C \text{ of } \mathcal{C} \text{ with } C \succ i \\ &= \text{distance from vertex } i \text{ to the root in } \text{tree}(\mathcal{S}) \\ &= \text{depth of vertex } i \text{ in } \text{tree}_{\square}(\mathcal{S}), \text{ and} \\ d_{\mathcal{S}}(ij) &= \# \text{ of chords } C \text{ of } \mathcal{C} \text{ with } C \succeq ij \\ &= \text{depth of cell } ij \text{ in } \text{tree}_{\square}(\mathcal{S}). \end{aligned}$$

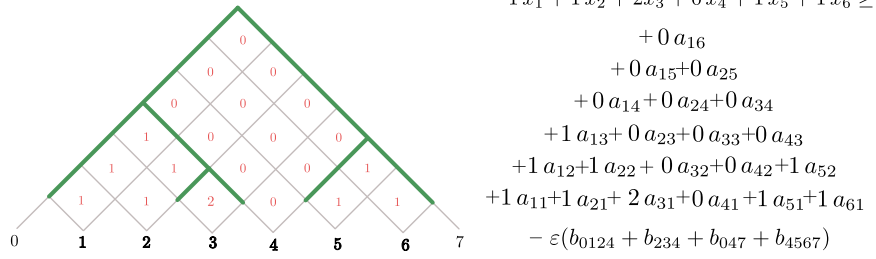
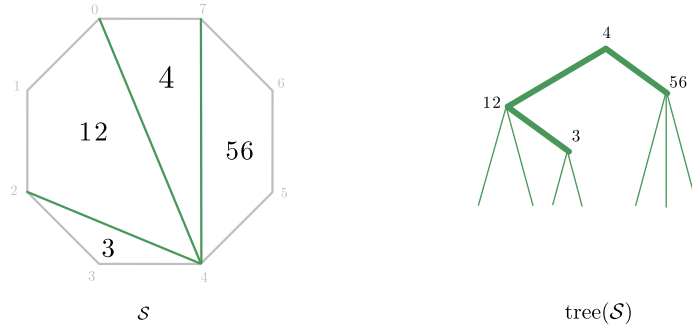
where $\text{tree}_{\square}(\mathcal{S})$ is the canonical embedding of the tree of \mathcal{S} in a square grid, and the depth of a vertex or a cell in $\text{tree}_{\square}(\mathcal{S})$ is the number of non-root vertices of the tree whose southwest/southeast shadow contains it; see Figure 16.

Theorem 4.13. *The cosmohedron $\text{Cosmo}_{n-1}(\mathbf{a}, \mathbf{b}) \subset \mathbb{R}^n$ is described by the irredundant system of inequalities*

$$\begin{aligned} \sum_{1 \leq i \leq n} x_i &= \sum_{ij \in \text{diag}(\diamond_{n+2})} a_{ij}, \\ \sum_{1 \leq i \leq n} d_{\mathcal{S}}(i) x_i &\geq \sum_{ij \in \text{diag}(\diamond_{n+2})} d_{\mathcal{S}}(ij) a_{ij} - \varepsilon \sum_{P \in \mathcal{S}} b(P) \text{ for any subdivision } \mathcal{S}. \end{aligned}$$

Proof. The lineality space of CosmoFan_{n-1} is $\mathbb{R}(1, \dots, 1)$, so the only equality satisfied by $\text{Cosmo}_{n-1}(\mathbf{a}, \mathbf{b})$ is the given one.

The irredundant inequalities of $\text{Cosmo}_{n-1}(\mathbf{a}, \mathbf{b})$ correspond to the rays of CosmoFan_{n-1} , which are in bijection with the minimal non-trivial Matryoshkas by Theorem 3.6. These are the subdivisions \mathcal{S} of \diamond_{n+2} – with no nesting – obtained from the non-crossing sets of chords \mathcal{C} . By (1), the ray $\mathbb{R}_{\geq 0}(\rho_{\mathcal{S}}) = \text{cone}(\mathcal{S})$ is given by the inequalities $x_v - x_w \geq 0$,



$$\begin{aligned}
& 1x_1 + 1x_2 + 2x_3 + 0x_4 + 1x_5 + 1x_6 \geq \\
& \quad + 0a_{16} \\
& \quad + 0a_{15} + 0a_{25} \\
& \quad + 0a_{14} + 0a_{24} + 0a_{34} \\
& \quad + 1a_{13} + 0a_{23} + 0a_{33} + 0a_{43} \\
& \quad + 1a_{12} + 1a_{22} + 0a_{32} + 0a_{42} + 1a_{52} \\
& \quad + 1a_{11} + 1a_{21} + 2a_{31} + 0a_{41} + 1a_{51} + 1a_{61} \\
& \quad - \varepsilon(b_{0124} + b_{234} + b_{047} + b_{4567})
\end{aligned}$$

Figure 16: A subdivision \mathcal{S} , its corresponding tree, its grid embedding with the corresponding cell depths, and the corresponding cosmo-hedral facet.

and hence $(\rho_{\mathcal{S}})_v - (\rho_{\mathcal{S}})_w = 1$, for all chords vw^\perp of \mathcal{S} . Therefore $\rho_{\mathcal{S}} = (d_{\mathcal{S}}(1), \dots, d_{\mathcal{S}}(n))$. Notice that $\rho_{\mathcal{S}} = \sum_{C \in \mathcal{C}} \rho_{\{C\}}$ and $\rho_{\{C\}} = e_{i+1} + \dots + e_{j-1}$ for $C = ij$.

To determine the extreme value of $(\rho_{\mathcal{S}}, x)$ for $x \in \text{Cosmo}_{n-1}(\mathbf{a}, \mathbf{b})$, we need to compute $(\rho_{\mathcal{S}}, c_M)$ for any maximal Matryoshka $M = (T, B)$ that refines \mathcal{S} ; see Figure 17. A chord C gets a label vw^\perp for a pair of adjacent triangles v, w in the triangulation T , and the definition in Lemma 3.3 gives $f_T(e_{vw}) = \rho_{\{C\}} + \mathbb{R}\mathbf{1}$; this is also a ray of the associahedral fan AssocFan_{n-1} . Therefore

$$\begin{aligned}
(\rho_{\mathcal{C}}, c_M) &= \sum_{C \in \mathcal{C}} (\rho_C, a_T + \varepsilon f^T(b_B)) \quad \text{by Definition 4.7,} \\
&= \sum_{C \in \mathcal{C}} (\rho_C, a_T) + \varepsilon \sum_{C=vw^\perp \in \mathcal{C}} (f_T(e_{vw}), f^T(b_B))
\end{aligned}$$

The first sum comes from the Loday associahedron; using (4) we obtain

$$\sum_{C \in \mathcal{C}} (\rho_C, a_T) = \sum_C \sum_{ij: ij \leq C} a_{ij} = \sum_{ij} d_C(ij) a_{ij}$$

Since f_T and f^T are dual, the second sum equals

$$\sum_{vw^\perp \in \mathcal{C}} (b_B)_{vw} = \sum_{vw^\perp \in \mathcal{C}} \left(b(\text{bracket}_B(vw)) - \sum_{tu \prec_B vw} b(\text{bracket}_B(tu)) \right)$$

using Lemma 4.6 for the vertices of the bracket associahedron. Here $tu \prec_B vw$ means that vw covers tu in the extended tree $^+(B)$ of the bracketing B .

Now, every diagonal d of the triangulation T that is not in \mathcal{C} is inside a polygon $P \in \mathcal{S}$, so $\text{bracket}_B(d) \subset P$ cannot contain any chord in \mathcal{C} . It follows that \mathcal{C} is an upper ideal of $\text{tree}^+(B)$, and that the brackets of the maximal non-elements of \mathcal{C} are precisely the polygons in \mathcal{S} . Also, $\text{tree}(B)$ has a unique maximal element $VW = \text{tree}(B)_{\max}$. Therefore

$$\begin{aligned} \sum_{vw^\perp \in \mathcal{C}} (b_B)_{vw} &= b(\text{bracket}_B(VW)) - \sum_{tu \in (\text{tree}(B) - \mathcal{C})_{\max}} b(\text{bracket}_B(tu)) \\ &= b(\diamond) - \sum_{P \in \mathcal{S}} b(P) \\ &= - \sum_{P \in \mathcal{S}} b(P) \end{aligned}$$

using that $b(\diamond) = 0$. The desired result follows. \square

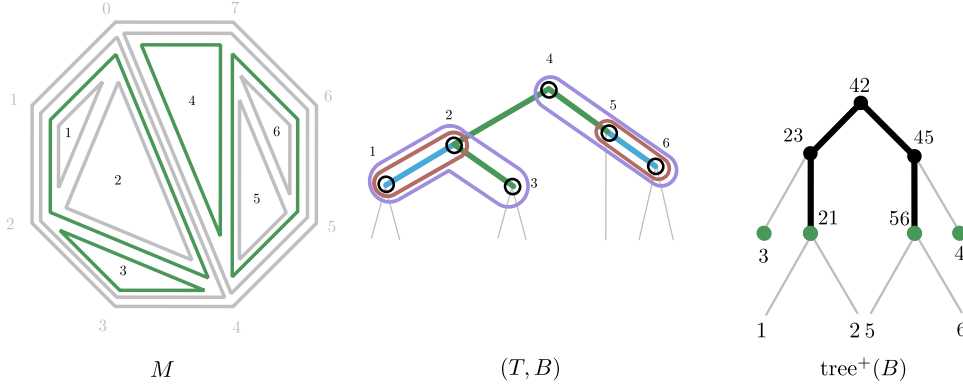


Figure 17: A vertex on the facet given by Figure 16.

Example 4.14. Figure 17 shows a Matryoshka M that refines the subdivision \mathcal{S} of Figure 16, so vertex c_M is on the facet that maximizes $\rho_{\mathcal{C}}$. In the computation of $(\rho_{\mathcal{C}}, c_M)$ above, the left sum is simply the dot product of the cell labels of $\text{tree}_{\square}(\mathcal{S})$ and \mathbf{a} . Writing $b(\cdot) = b(\text{bracket}_B(\cdot))$ for legibility, the right sum is

$$\begin{aligned} & [b(42) - b(23) - b(45)] + [b(23) - b(3) - b(21)] + [b(45) - b(56) - b(4)] \\ &= 0 - b(3) - b(21) - b(56) - b(4) \\ &= -b_{0124} - b_{234} - b_{047} - b_{4567} \end{aligned}$$

4.2.4 Isomorphism with the Arkani–Hamed–Figueiredo–Vazão cosmohedron.

Consider the kinematic space

$$\mathbb{R}^{\text{diag}(\diamond_{n+2})} = \{(X_{ij})_{ij \in \text{diag}(\diamond_{n+2})}\} \cong \mathbb{R}^{(n+2)(n-1)/2}$$

of assignments of a real number to each diagonal of \diamond_{n+2} . We assign $X_{0,n+1} = 0$ to the base edge $e = 0, n + 1$.

Let us fix:

- an arbitrary assignment $\mathbf{A} = (A_{ij} : 0 \leq i \ll j \leq n)$ of $\binom{n}{2}$ positive numbers to the diagonals of \diamond_{n+2} that do not involve $n + 1$, and
- a small convex vector $\mathbf{B} = (B(P) : P \text{ is a subpolygon of } \diamond_{n+2})$.

Note that the vector \mathbf{b} of the previous section was concave, and B is convex.

Arkani-Hamed, Bai, He, and Yan [AHBHY18b] defined the *associahedron*³ $\text{ABHY}_{n-1}(\mathbf{A})$ to be the $(n - 1)$ -dimensional polytope

$$\{X \in \mathbb{R}^{\text{diag}(\diamond_{n+2})} : X_{r,s} + X_{r+1,s+1} - X_{r,s+1} - X_{r+1,s} = A_{rs} \text{ for } 0 \leq r \ll s \leq n \\ X_{r,s} \geq 0 \text{ for } 0 \leq r \ll s \leq n + 1, \}$$
 (9)

where $X_{0,n+1} = X_{r,r+1} := 0$ for $0 \leq r \leq n$, and again, $r \ll s$ means that $s - r \geq 2$.

Arkani-Hamed, Figueiredo, and Vazão [AHFV25] defined the following polytope, and conjectured that its face structure matches the combinatorics of Matryoshkas.

Definition 4.15. *The cosmohedron $\text{AFV}_{n-1}(\mathbf{A}, \mathbf{B})$ is the $(n - 1)$ -dimensional polytope*

$$\{X \in \mathbb{R}^{\text{diag}(\diamond_{n+2})} : X_{r,s} \geq 0 \text{ for } 0 \leq r \ll s \leq n + 1, \\ X_{r,s} + X_{r+1,s+1} - X_{r,s+1} - X_{r+1,s} = A_{rs} \text{ for } 0 \leq r \ll s \leq n, \\ \sum_{C \in \mathcal{C}} X_C \geq \sum_{P \in \mathcal{S}(C)} B(P) : \text{ for non-crossing } \mathcal{C} \subset \text{diag}(\diamond_{n+2}) \}$$
 (10)

where $\mathcal{S}(C)$ is the set of polygons in the subdivision of \diamond_{n+2} induced by C .

Theorem 4.16. *The map $X_{r,s} = \sum_{r < i < s} x_i - \sum_{r \leq i < j \leq s} a_{ij}$ gives linear isomorphisms*

1. *between the associahedra $\text{Loday}_{n-1}(\mathbf{a})$ and $\text{ABHY}_{n-1}(\mathbf{A})$, and*
2. *between the cosmohedra $\text{Cosmo}_{n-1}(\mathbf{a}, \mathbf{b})$ and $\text{AFV}_{n-1}(\mathbf{A}, \mathbf{B})$,*

where $A_{ij} = a_{i,j+1}$ for each diagonal $ij \in \text{diag}(\diamond_{n+2})$ and $B(P) = -\varepsilon b(P)$ for sufficiently small $\varepsilon > 0$.

Before proving this, we note that \mathbf{a} has size $\binom{n+1}{2}$ whereas \mathbf{A} has size $\binom{n}{2}$; this is because the n entries $a_{i-1,i+1}$ do not enter the definition of $\text{AFV}_{n-1}(\mathbf{A}, \mathbf{B})$. These entries are the coefficients of the points $\Delta_{\{i\}} = e_i$ in the Minkowski sum of $\text{Loday}_{n-1}(\mathbf{a})$, so they only cause a translation of the polytope.

Proof of Theorem 4.16. 1. A vector $x \in \mathbb{R}^n$ determines a vector X in kinematic space that satisfies the linearly independent equations

$$X_{r,s} + X_{r+1,s+1} - X_{r,s+1} - X_{r+1,s} = a_{r,s+1} = A_{r,s} \quad \text{for } 0 \leq r \ll s \leq n.$$

³In [AHBHY18b] this is denoted \mathcal{A}_{n+2} . We use the subscript $n - 1$ to match the root system A_{n-1} and the dimension of the polytope.

as one readily verifies. These $\binom{n}{2}$ equations restrict X to a vector space of dimension $(n+2)(n-1)/2 - \binom{n}{2} = n-1$, which is precisely the dimension of the associahedron. The inequalities $X_{r-1,s+1} \geq 0$ are restatements of the defining inequalities of $\text{Loday}_{n-1}(\mathbf{a})$ as written in (4), proving the desired isomorphism.

2. It remains to translate the inequalities of the cosmohedron $\text{Cosmo}_{n-1}(\mathbf{a}, \mathbf{b})$:

$$\sum_{1 \leq i \leq n} d_{\mathcal{C}}(i) x_i \geq \sum_{ij \in \text{diag}(\hat{\diamond}_{n+2})} d_{\mathcal{C}}(ij) a_{ij} - \varepsilon \sum_{P \in \mathcal{S}} b(P)$$

for any subdivision \mathcal{S} with chords \mathcal{C} . We rewrite this as

$$\sum_{C \in \mathcal{C}} \left(\sum_{1 \leq i \leq n} d_{\{C\}}(i) x_i \right) \geq \sum_{C \in \mathcal{C}} \left(\sum_{ij \in \text{diag}(\hat{\diamond}_{n+2})} d_{\{C\}}(ij) a_{ij} \right) - \varepsilon \sum_{P \in \mathcal{S}} b(P).$$

Now notice that $d_{\{C\}}(i)$ is 1 if $C \succ i$ and 0 otherwise, and $d_{\{C\}}(ij)$ is 1 if $C \succeq ij$ and 0 otherwise, so

$$X_C = \sum_{1 \leq i \leq n} d_{\{C\}}(i) x_i - \sum_{ij \in \text{diag}(\hat{\diamond}_{n+2})} d_{\{C\}}(ij) a_{ij}.$$

Thus the inequalities of $\text{Cosmo}_{n-1}(\mathbf{a}, \mathbf{b})$ in x -space are equivalent to the inequalities of $\text{AFV}_{n-1}(\mathbf{A}, \mathbf{B})$ in X -space. We note that \mathbf{B} is concave if and only if \mathbf{b} is concave, and \mathbf{B} is made sufficiently small – as [AHFV25] requires – by making ε sufficiently small. \square

In particular, this proves one of our main results: the correctness of Arkani-Hamed, Figueiredo, and Vazão’s construction.

Corollary 4.17. *The face poset of Arkani-Hamed, Figueiredo, and Vazão’s cosmohedron AFV_{n-1} is anti-isomorphic to the poset of Matryoshkas of an $(n+2)$ -gon.*

It also lets us determine precisely what their requirement of a “small enough” B means. Theorem 4.11 proves that $\text{Cosmo}_{n-1}(\mathbf{a}, \mathbf{b})$ is a cosmohedron for $0 < \varepsilon < (\min a_{ij})/24(\max b(P))$; it would be interesting to trace the proof carefully and find the optimal constant.

5 Face structure and enumeration

The dimension of a face. Recall that $M_{\text{not min}}$ is the number of polygons in a Matryoshka M that are not inclusion-minimal.

Lemma 5.1. *The cone of a Matryoshka M in the cosmoedral fan has dimension*

$$\dim \text{cone}(M) = |M_{\text{not min}}| + 1.$$

Proof. If (T, B) is the bracketed tree corresponding to M from Lemma 2.10, we have $\text{cone}(M) = \text{cone}(T, B) = f_T(\text{cone}_T(B)_{\geq 0})$ and its dimension equals the dimension of $\text{cone}_T(B)$, which is the number of brackets in B . This equals the number of non-minimal polygons of the Matryoshka M . \square

Faces of the cosmohedron factor combinatorially, but not geometrically. The faces of the cosmohedron have an interesting factorization property that is subtler and weaker than the analogous property for bracket associahedra.

Bracket associahedra⁴ have the property that any facet factors canonically as a Cartesian product of two bracket associahedra, as we now describe. Let H be a graph, β be a bracket of H , and H/β the contraction of β in H , whose edges are in bijection with $E(H) - E(\beta)$. Also let $b|_{\beta}(\beta') = b(\beta')$ for each bracket β' of β and $b/\beta(\beta'') = b(\beta \cup \beta'') - b(\beta)$ for each bracket β'' of H/β . Note that if b is concave on $\text{Brackets}(H)$, then $b|_{\beta}$ and b/β are concave on $\text{Brackets}(\beta)$ and $\text{Brackets}(H/\beta)$, respectively.

Lemma 5.2. [AA23] *For any graph H and any bracket β of H , the e_{β} -maximal facet of $\text{BrAssoc}_H(\mathbf{b})$ is linearly isomorphic to the product of bracket associahedra*

$$(\text{BrAssoc}_H(\mathbf{b}))_{\beta} \cong \text{BrAssoc}_{\beta}(\mathbf{b}|_{\beta}) \times \text{BrAssoc}_{H/\beta}(\mathbf{b}/\beta) \subset \mathbb{R}^{E(\beta)} \times \mathbb{R}^{E(H/\beta)}$$

Facets, and hence faces, of cosmohedra have an analogous factorization property – but it is significantly weaker. The first statement below was observed in [AHFV25], but it holds an unexpected surprise.

Proposition 5.3. *Let \mathcal{S} be a subdivision of the $(n + 2)$ – gon into polygons P_1, \dots, P_k where P_i is an $(n_i + 2)$ -gon. Let $T(\mathcal{S})$ be its dual graph, which is a tree. Then the facet of Cosmo_n corresponding to \mathcal{S} is **combinatorially** isomorphic to the product*

$$(\text{Cosmo}_n)_{\mathcal{S}} \cong^{\text{comb}} \text{BrAssoc}_{T(\mathcal{S})} \times (\text{Cosmo}_{n_1}) \times \dots \times (\text{Cosmo}_{n_k}).$$

*However, this factorization **does not hold geometrically**.*

Proof. The face poset of $L(\text{Cosmo}_n)$ is the opposite of the the poset M_n of Matryoshkas of the $(n + 2)$ -gon. The interval $[\hat{0}, (\text{Cosmo}_n)_{\mathcal{S}}]$ corresponds to the subposet $(M_n)_{\geq \mathcal{S}}$ of Matryoshkas refining \mathcal{S} . Such a Matryoshka is obtained by making $k + 1$ independent choices:

(i) refining each of the k polygons P_i internally with its own Matryoshka – the ways of doing this are in order-reversing bijection with the faces of Cosmo_{n_i} – and

(ii) refining \mathcal{S} externally with compatible polygons that are unions of the P_i s – the ways of doing this are in order-reversing bijection with the bracketings of $T(\mathcal{S})$.

These independent choices are compatible with the order relations of the posets, giving the desired result. \square

The above isomorphism is not alinear isomorphism of polyhedra in general. We can already see this in the cosmohedron Cosmo_3 . Consider the subdivision of a hexagon by a long and a short diagonal, as illustrated in Figure 18. Proposition 5.3 predicts that this facet is combinatorially isomorphic to the product $T(P_2) \times \text{Cosmo}_2 \times \text{Cosmo}_1 \times \text{Cosmo}_1 =$

⁴and, more broadly, generalized permutahedra [AA23, Edm70]

$|\times|\times\cdot\times\cdot$ which is a quadrilateral. We claim that this face is a trapezoid; we can see this in Figure 2, but let us verify it algebraically. We use Proposition 4.8 to compute:

$$\begin{aligned} c_{M_1} &= (1, 2, 6, 1) + \varepsilon(0, -3, 3, 0) & c_{N_1} &= (2, 1, 6, 1) + \varepsilon(-3, 0, 3, 0) \\ c_{M_2} &= (2, 1, 6, 1) + \varepsilon(-1, 0, 3, -2) & c_{N_2} &= (1, 2, 6, 1) + \varepsilon(0, -1, 3, -2) \end{aligned}$$

and notice that the opposite sides

$$c_{M_1} - c_{N_1} = (1 - 3\varepsilon)(e_2 - e_1), \quad c_{M_2} - c_{N_2} = (1 - \varepsilon)(e_1 - e_2),$$

are parallel but not equal. Since it is not a parallelogram, this quadrilateral face cannot be expressed as a Cartesian product of two edges.

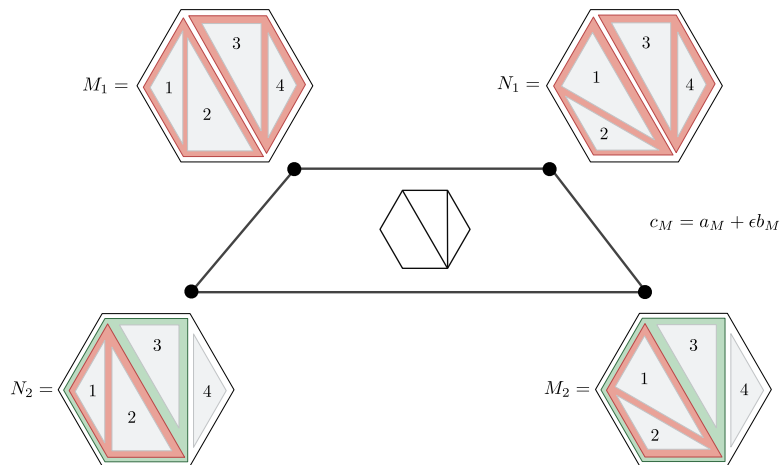


Figure 18: A trapezoidal face of the cosmohedron.

5.1 Enumeration

Having understood the combinatorial structure of the cosmohedron, we are now ready to enumerate its faces. We also compute the face numbers of the *correlatron* of [FV26], which are closely related in an elegant and surprising way.

5.2 Counting the faces of the cosmohedron

Our next goal is to compute the number of faces of the cosmohedron in each dimension. We list these numbers in Table 1 for $n \geq 6$.

It is useful to organize them in the f -polynomial $f_n(t) = f_{\text{AssocFan}_{n-1}}(t)$ and the modified f -polynomial $g_n(t) = [(1+t)f_n(t) - 1]/t$, whose coefficients are the consecutive pair sums of the coefficients of $f_n(t)$:

$$\begin{aligned} f_1(t) &= 1, & f_2(t) &= 1 + 2t, & f_3(t) &= 1 + 10t + 10t^2, & f_4(t) &= 1 + 44t + 114t^2 + 72t^3, \dots \\ g_1(t) &= 1, & g_2(t) &= 3 + 2t, & g_3(t) &= 11 + 20t + 10t^2, & g_4(t) &= 45 + 148t + 186t^2 + 72t^3, \dots \end{aligned}$$

cosmohedron	0	1	2	3	4	5	6
0	1						
1	1	2					
2	1	10	10				
3	1	44	114	72			
4	1	196	952	1400	644		
5	1	902	7116	18040	18528	6704	
6	1	4278	50550	194616	332664	262728	78408

Table 1: The number of d -codimensional faces of the n th cosmohedron on $n + 2$ points.

Theorem 5.4. *The f -polynomials of the cosmohedral fans form the unique sequence $\{f_n(t)\}_{n \geq 1}$ of polynomials such that the power series in $(\mathbb{R}[t])[x]$*

$$\begin{aligned} x - f_1(t)x^2 - f_2(t)x^3 - f_3(t)x^4 - \dots \\ x + g_1(t)x^2 + g_2(t)x^3 + g_3(t)x^4 + \dots \end{aligned}$$

are compositional inverses, where $g_n(t) = [(1+t)f_n(t) - 1]/t$. Equivalently, the power series $A_t(x) = x - f_1(t)x^2 - f_2(t)x^3 + \dots$ satisfies

$$A_t \left(\frac{(2t+1)x - (2t+2)x^2}{t(1-x)} - \left(1 + \frac{1}{t}\right) A_t(x) \right) = x.$$

Proof. Every face F of the cosmohedral fan is contained in a unique minimal face of the associahedral fan $A(F)$. If F corresponds to a Matryoshka M on an $(n+2)$ -gon then $A(F)$ corresponds to the subdivision M_{max} given by the maximal polygons P_1, \dots, P_k of M . If P_i is an (n_i+2) -gon then $n = n_1 + \dots + n_k$.

Conversely, consider a face A of the associahedral fan, corresponding to a non-trivial subdivision \mathcal{S} of the $(n+2)$ -gon into an (n_i+2) -gon P_i for $1 \leq i \leq k$. The faces F of the cosmohedron such that $A(F) = A$ correspond to the Matryoshkas with $M_{max} = \mathcal{S}$; these are obtained by choosing arbitrary Matryoshkas M_1, \dots, M_k on P_1, \dots, P_k , that is, faces F_1, \dots, F_k on the corresponding cosmohedral fans. Furthermore, we have

$$\dim F = 1 + |M_{\text{not min}}| = 1 + \sum_{i=1}^k (|(M_i)_{\text{not min}}| + 1) = 1 + \sum_{i=1}^k \dim F_i.$$

so the f -numbers of the cosmohedron Cosmo_{n-1} are given by the recurrence

$$f_n(t) = \sum_{F \leq \text{Cosmo}_{n-1}} t^{\text{codim } F} = 1 + t \sum_{A \leq \text{Assoc}_{n-1}} f_{n_1}(t) \cdots f_{n_k}(t). \quad (11)$$

Adding the full face Assoc_{n-1} to both sides, we obtain

$$g_n(t) = \sum_{A \leq \text{Assoc}_{n-1}} f_A(t) \quad (12)$$

using the notation of [AA23]: if x_1, x_2, \dots , is a sequence and a face A of the associahedron Assoc_{n-1} is combinatorially a product of associahedra $\text{Assoc}_{n_1-1} \times \dots \times \text{Assoc}_{n_k-1}$, then we write $x_A = x_{n_1} \cdots x_{n_k}$.

Arriving at (12) is a pleasant déjà vu: the associahedral version of Lagrange inversion obtained in [AA23, Theorem 2.4.3] tells us that this recurrence is precisely equivalent to the assertion that $x - f_1(t)x^2 - f_2(t)x^3 - \dots$ and $x + g_1(t)x^2 + g_2(t)x^3 + \dots$ are compositional inverses.

The second formulation follows from the first by a straightforward computation. \square

A pleasant feature of this proof is that Aguiar and Ardila discovered the associahedral version of Lagrange inversion in [AA23, Theorem 2.4.3] within the Hopf-algebraic structure of Loday's associahedra, specifically. Is it coincidental that this is the same associahedron in our construction? Might there a deeper underlying algebraic explanation? Cosmohedra feature an operadic structure that is the subject of an upcoming paper.

Corollary 5.5. *The f -polynomials of the cosmohedral fans are given by the recurrence $f_1(t) = 1$ and*

$$f_n(t) = 1 + t \sum_{1^{a_1} 2^{a_2} \dots \vdash n} \frac{(n + a_1 + a_2 + \dots)!}{(n + 1)!} \cdot \frac{f_1(t)^{a_1}}{a_1!} \frac{f_2(t)^{a_2}}{a_2!} \dots \quad \text{for } n \geq 2.$$

summing over the non-trivial partitions of n as a sum $n = a_1 \cdot 1 + a_2 \cdot 2 + \dots + a_{n-1} \cdot (n-1)$.

Proof. This is a reformulation of the recurrence formula (11), taking into account that the number of subdivisions of an $(n+2)$ -gon into a_2 triangles, a_3 quadrilaterals, etc. is $(n + a_1 + a_2 + \dots)! / (n + 1)! a_1! a_2! \dots$ [Sta99, Theorem 5.3.10]. \square

5.3 Asymptotic growth

Facets. The facets of the cosmohedron Cosmo_{n-1} are in bijection with the non-trivial subdivisions of an $(n+2)$ -gon, or equivalently the non-trivial parenthesizations of a string of $n+1$ letters. Their number, listed in column 1 of Table 1, is one less than the n -th *Hipparchus–Schröder number*. This is one of the oldest known combinatorial quantities, studied by astronomer and mathematician Hipparchus in the second century B.C.; for a fascinating historical account see [Sta99, Sta97]. These numbers are sequence A001003 in the Online Encyclopedia of Integer Sequences; their generating function is

$$x + x^2 + 3x^3 + 11x^4 + 45x^5 + 197x^6 + \dots = \frac{1}{4} \left(1 + x - \sqrt{1 - 6x + x^2} \right)$$

The asymptotic growth of the sequence is controlled by $3 + \sqrt{8}$, the inverse of the smallest root of $1 - 6x + x^2$. More precisely [Knu97], the number r_n of rays grows like

$$r_n \sim \frac{1}{4} \sqrt{(\sqrt{18} - 4)/\pi} \cdot (3 + \sqrt{8})^n \cdot n^{-3/2}.$$

Vertices. The vertices of the cosmohedron Cosmo_{n-1} are in bijection with the maximal Matryoshkas on an $(n+2)$ -gon. The following lemma is stated in [AHFV25] with a minor typo; we include a proof.

Lemma 5.6. *The number m_n of maximal Matryoshkas on an $(n+2)$ -gon is given by the recurrence relation*

$$m_1 = 1, \quad m_n = \sum_{k=1}^{n-1} (k+1)m_k m_{n-k} \text{ for } n \geq 2.$$

The formal power series $M(x) = \sum_{n \geq 2} m_{n-1} x^n = x^2 + 2x^3 + 10x^4 + 72x^5 + \dots$ is given by the differential equation

$$M'(x) = 1 - \frac{x^2}{M(x)}.$$

Proof. Fix an edge e of the $(n+2)$ -gon. A maximal Matryoshka is obtained by splitting an $(n+2)$ -gon into a $(k+2)$ -gon that contains e and an $(n-k+2)$ -gon, and then putting a maximal Matryoshka in each one of them. There are $k+1$ possible positions for the outer $(k+2)$ -gon that contains e , and m_k and m_{n-k} choices for the two inner Matryoshkas, respectively. The differential equation follows by a straightforward computation. \square

In particular this means that $M(x)$ is a D -algebraic series; that is, its derivatives satisfy a polynomial equation with coefficients in $\mathbb{C}[x]$; in fact one of the simplest such equations: $M(x) = x^2 + M(x)M'(x)$. Compare this with the algebraic power series for Catalan numbers, which satisfies one of the simplest algebraic equations $C(x) = 1 + xC(x)^2$.

Melczer [Mel21] writes that “*D-algebraic series seem to be on the border between decidability and undecidability, perhaps tipped to the side of undecidability. [...] Compared to D-finite functions not much is known about D-algebraic asymptotics, and in general not much can be said.*”. Since $M(x)$ is one of the simplest D -algebraic series, the following conjecture based on numerical evidence, is an interesting challenge.

Conjecture 5.7. *The number of Matryoshkas grows asymptotically like*

$$m_n \sim c \cdot n! \cdot n^4$$

for $c = 0.005428\dots$

All faces. A more ambitious goal would be to find the asymptotics for the total number of faces, or even the limit distribution of the normalized f -vector. We do not know how to do this.

5.4 Counting the faces of the correlatron

Another object that naturally arises in physics, and that is intimately related to the cosmodhedron, is the *correlatron* introduced by Figueiredo and Vazão in [FV26]. This polytope captures the combinatorics of the correlator, which is the physical observable that one can compute from the wavefunction. The combinatorial structure of this

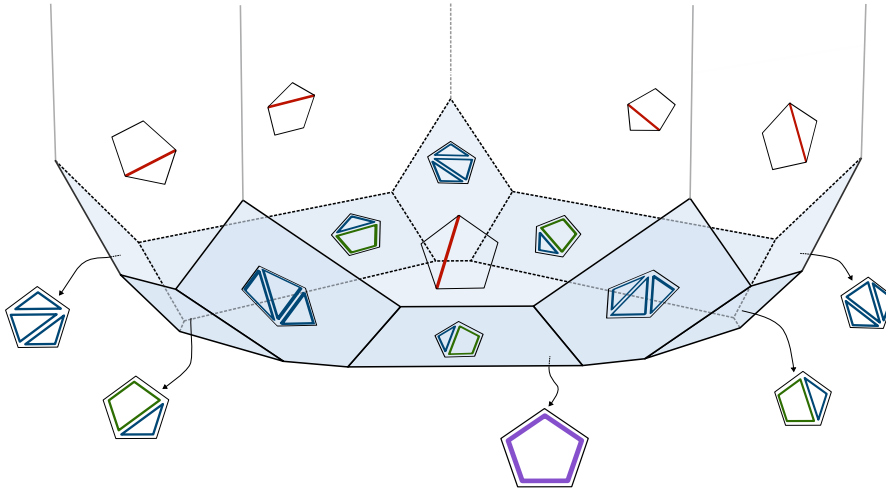


Figure 19: The three-dimensional correlatron.

object is described by the interacting combinatorics of chords (the combinatorics of associahedra) and subpolygons (the combinatorics of cosmohedra).

The correlatron is an unbounded n -dimensional polyhedron, with faces of two kinds:

1. The bounded faces of the correlatron are labelled by the compatible pairs (\mathcal{C}, M) , consisting of a set \mathcal{C} of non-crossing chords and a Matryoshka M on an $(n+2)$ -gon. Here \mathcal{C} and M are *compatible* if no chord in \mathcal{C} intersects a polygon in M ; or equivalently, if \mathcal{C} is a subset of the diagonals in the underlying subdivision M_{\max} of M . The bounded face inclusions of the correlatron are given by the reverse inclusion of chords and Matryoshkas:

$$\{\mathcal{C}', M'\} \text{ is a face of } \{\mathcal{C}, M\} \text{ in the correlatron if and only if } \mathcal{C} \subseteq \mathcal{C}' \text{ and } M \leq M'.$$

2. The unbounded faces are labeled by chords $\underline{\mathcal{C}}$, which we underline to distinguish them easily from the earlier faces. An unbounded face $\underline{\mathcal{C}}$ is contained in $\underline{\mathcal{C}'}$ if and only if $\mathcal{C} \subseteq \mathcal{C}'$. An unbounded face $\underline{\mathcal{C}}$ also contains the bounded faces (\mathcal{C}', M') such that $\mathcal{C} \subseteq \mathcal{C}'$.

Note in particular that the correlatron has a bottom cosmohedral facet Cosmo_{n-1} corresponding to the pairs of the form (\emptyset, M) . Its unbounded directions look like a cone over Assoc_{n-1} . Since the unbounded faces of the correlatron are enumerated by the well-known f -vector of the associahedron, we focus on enumerating the bounded part.

Theorem 5.8. *The reverse f -polynomials of the bounded correlatrons $h_1(t) = t, h_2(t) = 3t+4t^2, h_3(t) = 11t+35t^2+25t^3, \dots$ form the unique sequence $\{h_n(t)\}_{n \geq 1}$ of polynomials such that the power series in $(\mathbb{R}[t])[x]$*

$$\begin{aligned} x + h_1(t)x^2 + h_2(t)x^3 + h_3(t)x^4 + \dots \\ x - tg_1(t)x^2 - tg_2(t)x^3 - tg_3(t)x^4 + \dots \end{aligned}$$

are compositional inverses, where $g_n(t) = [(1+t)f_n(t)-1]/t$ and f_n is the f -polynomial of the n th cosmohedral fan. Equivalently, the power series $H_t(x) = x+h_1(t)x^2+h_2(t)x^3+\dots$

correlatron	0	1	2	3	4	5	6
0	0						
1	0	1					
2	0	3	4				
3	0	11	35	25			
4	0	45	251	405	200		
5	0	197	1694	4592	4984	1890	
6	0	903	11158	44932	80036	65606	20248

Table 2: The number of d -codimensional faces of the n th bounded correlatron.

satisfies

$$H_t \left(\frac{(t+1)x^2 - tx}{1-x} + (t+1)A_t(x) \right) = x$$

where $A_t(x)$ is the generating function for the f -vectors of cosmohedra in Theorem 5.4.

Proof. Every face F of the truncated correlatron corresponds to a pair (\mathcal{C}, M) where M is non-trivial. Let \mathcal{C} create a subdivision into polygons P_1, \dots, P_k , and let $A(F)$ be the corresponding face of the associahedral fan corresponding to the set of chords \mathcal{C} . A Matryoshka M compatible with \mathcal{C} restricts to Matryoshkas M_1, \dots, M_k on P_1, \dots, P_k . One can almost recover M from the M_i s: the only additional information is whether M contains the polygon P_i (in which case we let $\varepsilon_i = 1$) or it does not (in which case $\varepsilon_i = 0$). We note that if M_i is the trivial Matryoshka on P_i then $\varepsilon_i = 1$ necessarily; if M_i is non-trivial then ε_i can be chosen freely. Finally notice that

$$\text{codim } F = |\mathcal{C}| + |M_{\text{non-min}}| = k + \sum_{i=1}^k |(P_i)_{\text{non-min}}| + \sum_{i=1}^k (\varepsilon_i - 1)$$

Therefore

$$\begin{aligned}
h_n(t) &= \sum_{F \leq \text{BdCorrela}_n} t^{\text{codim } F} \\
&= \sum_{\substack{A \leq \text{Assoc}_n \\ S(A) = \{P_1, \dots, P_k\}}} t^k \sum_{\substack{M_1, \dots, M_k \\ \varepsilon_1, \dots, \varepsilon_k}} \left(\prod_{i=1}^k t^{|(P_i)_{\text{non-min}}| + \varepsilon_i - 1} \right) \\
&= \sum_{A \leq \text{Assoc}_n} t^{|\mathcal{C}|} \prod_{i=1}^k \left(1 + \sum_{M_i \neq \{P_i\}} t^{|(P_i)_{\text{non-min}}|} \left(1 + \frac{1}{t} \right) \right) \\
&= \sum_{A \leq \text{Assoc}_n} t^{|\mathcal{C}|} \prod_{i=1}^k \left(1 + (f_{P_i}(t) - 1) \left(1 + \frac{1}{t} \right) \right) \\
&= \sum_{A \leq \text{Assoc}_n} t \prod_{i=1}^k ((1+t)f_{P_i}(t) - 1) \\
&= \sum_{A \leq \text{Assoc}_n} t g_A(t),
\end{aligned}$$

again using the notation of [AA23]. The associahedral version of Lagrange inversion [AA23, Theorem 2.4.3] then tells us that $x - tg_1(t)x^2 - tg_2(t)x^3 - tg_3(t)x^4 + \dots$ and $x + h_1(t)x^2 + h_2(t)x^3 + h_3(t)x^3 + \dots$ are compositional inverses.

The second formulation follows from a straightforward computation. \square

Note the remarkable similarity between the formulas in Theorems 5.4 and 5.8 for the f -vectors of the cosmohedron and bounded correlatron. Is this an indication of something deeper?

6 Elusive qualities of the cosmohedron

The cosmohedron is related to several beautiful polytopes in the literature, either through direct connections or indirect analogies. These include the permutahedron [BS13, Sch13], the associahedron [Lod04, Sta63], graph associahedra [CD06], the nested permutahedron [CL22], the permutoassociahedron [CL23], and the permutonestohedron [Gai15], among others.

However, there are numerous ways in which the combinatorial structure of the cosmohedron is more subtle than its predecessors. In turn, this makes it significantly harder to prove the correctness of this construction. In this section we discuss some of the intricacies that arise – and that remain in future polytopes of interest.

The cosmohedron is not very symmetric. The permutahedron, the nested permutahedron, the permuto-associahedron, and the permutonestohedron are symmetric under the natural action of S_n onto \mathbb{R}^n . This allows us to express them as the convex hull of the S_n -orbit

of a point, or of a carefully placed permutahedron or associahedron, respectively. The cosmohedron only has a natural action of the dihedral group.

The cosmohedron is made of an associahedron's worth of different bracket associahedra. It requires much greater care to choose a suitable realization of each bracket associahedron that are compatible with each other, and then place each one of them at the correct position, in order to make sure that they glue correctly into a well-behaved global polytope. This is a subtle matter.

The cosmohedron is not simplicial. This is not a surprise; none of the polytopes on the above list are simplicial. Proposition 5.3 shows that the only simplicial faces of the cosmohedron are the vertices and edges.

The cosmohedron is not simple. This is more surprising, because all the other polytopes in the list above are simple. The only simple vertices of the cosmohedron are those whose Matryoshka is nested linearly. To see this, let $M = (T, B)$ be a maximal Matryoshka. The facets of $\text{cone}(M)$ correspond to the facets of $\text{cone}_T(B)_{\geq 0}$. These in turn correspond to the $n - 2$ facets of $\text{cone}_T(B)$ (which is simplicial), and the facets of the form $y_e \geq 0$ for a minimal bracket in B , which is an edge in B , and corresponds to a quadrilateral in M . Therefore

$$(\text{number of facets of } \text{cone}(M)) = n - 2 + (\text{number of quadrilaterals in } M).$$

This can be any number between $n - 1$ and $\lfloor 3n/2 \rfloor - 2$. It equals $n - 1$ precisely if M has exactly one quadrilateral, which happens precisely when M is linearly nested.

Why is this a challenge? The fact that polytopes like (bracket) associahedra are simple allows for a flexible construction. One can start with an easy polytope (in these cases a simplex), and then create new facets one at a time by making cuts that are “not too deep”, so that the facets meet transversally. The required care only involves depth inequalities; see [Lod04, Dev09, PPP23].

We **do** construct the cosmohedron through a chiseling of faces of the associahedron, but we have to be much more careful now: the chiseling has to line up perfectly to create the required non-transversal crossings. The required care now involves many equalities as well as inequalities. In AFV_{n-1} , these equalities manifest in the fact that the linear forms $\sum_{C \in \mathcal{C}} X_C$ satisfy many linear relations between them. In Cosmo_{n-1} , they manifest in the many requirements $b_B = b_{B'}$ in Remark 4.12.

The cosmohedron depends delicately on the chiseling. To further illustrate the point above, recall that in our construction of the cosmohedron, we chiseled a Devadoss bracket associahedron at each vertex of the associahedron. This choice is important. For instance, despite their usefulness in other settings, the Postnikov bracket associahedra do not produce the cosmohedron upon chiseling.

We don't know the cosmohedron to be a deformation of an easy polytope. The *deformations of the permutahedron*, also known as *generalized permutahedra* or *polymatroids*, are obtained from the permutahedron by moving facets without pushing them past

vertices. They are characterized by having the same edge directions as the permutahedron, namely $e_i - e_j$ for $i \neq j$ [ACEP20, Prop. 2.6], [Pos09].

Generalizing this setup, Castillo and Liu [CL22] defined the *nested permutahedron* $\text{Perm}(\alpha, \beta)$ by placing an $(n-1)$ -permutahedron at each corner of the n -permutahedron. They studied its deformations, and showed that Kapranov’s *permutoassociahedron* [Kap93] – which has an $(n-1)$ -associahedron at each vertex of the n -permutahedron, can be obtained as a deformation of $\text{Perm}(\alpha, \beta)$.

The cosmohedron would seem to fit in the same framework; our proof of Theorem 3.6 even shows that the cosmohedron has the same edge directions as $\text{Perm}(\alpha, \beta)$, namely $e_i - e_j$ and $e_i - e_j + e_k - e_l$ for distinct i, j, k, l . This makes it natural to wonder: Is our cosmohedron a deformation of the nested permutahedron? **It is not.**

Dually, we claim that our cosmohedral fan is not a coarsening of the nested braid fan Br_n^2 of Castillo and Liu [CL22]. The $n! \cdot (n-1)!$ facets of Br_n^2 correspond to the choices of an order $\sigma \in S_n$ of the coordinates of $x \in \mathbb{R}^n$ and an order $\tau \in S_{n-1}$ of their consecutive differences. For example, the set of points $x \in \mathbb{R}^4$ with

$$x_3 > x_1 > x_4 > x_2 \quad \text{and} \quad x_3 - x_1 > x_1 - x_4 > x_4 - x_2 > 0$$

is a maximal open cone of Br_4^2 . Such a point must satisfy two inequalities $x_3 - x_4 > x_3 - x_1$ and $x_1 - x_2 > 0$ of the Matryoshkal cone(M_2) of Figure 9, but it may or may not satisfy the third inequality $x_3 - x_1 > x_1 - x_2$. Therefore this maximal cone of Br_4^2 intersects cone(M_2) full-dimensionally, but is not contained in it. This proves our claim.

Why is this a challenge? Generalized permutahedra are in bijection with submodular functions on the subsets of $[n]$. Thus to construct them, it is sufficient to choose the correct submodular function, and to use the theory of polymatroids and generalized permutahedra to prove the correct combinatorial structure; see [AA23, Pos09].

We do not know an analog to the permutahedron for this setting. It would be very interesting to find a nice polytope such that (i) it can be deformed to obtained the cosmohedron, (ii) its deformation cone can be nicely described combinatorially, and (iii) the face structure of its deformations are combinatorially well-behaved.

We don’t know the cosmohedron to be a Minkowski sum of easy polytopes. The permutahedron, Loday’s associahedron, and Postnikov’s graph abd bracket associahedra can all be expressed as Minkowski sums of faces of the standard simplex. This gives a simple proof that their face structure – which is the common refinement of the easy face structures of the summands – is correct. For details see [Pos09].

We do not know how to do this in our setting. It would be interesting to find a realization of the cosmohedron that can be decomposed easily and usefully as a Minkowski sum.

The faces of the cosmohedron only factor combinatorially. As discussed in Lemma 5.2, faces of a generalized permutahedra factor into smaller generalized permutahedra [AA23, Edm70]. For permutahedra, associahedra, and graph associahedra, any facet of a polytope in these families is a product of smaller polytopes in the same family. This gives an inductive strategy to prove their combinatorial structure, introduced for associahedra in [MSS02]. One simply needs to guarantee that the facets are placed correctly; again,

because these polytopes are simple, this step only requires verifying the appropriate depth inequalities. Once the facets are laid out correctly, thanks to their factorization, the rest of the face structure can be proved recursively.

In our realization of the cosmohedron, each one of the facets is **combinatorially, but not geometrically** a product of cosmohedra and a bracket associahedron. Therefore its combinatorial structure cannot be proved using this inductive strategy. It would be interesting to find a realization of the cosmohedron where the face factorization of Proposition 5.3 holds geometrically.

Outlook: chiseled polytopes. In an upcoming paper we present a general family of \mathcal{X} in Y polytopes that contains all of the polytopes above, and numerous other polytopes of interest, old and new. Our proof strategy for the combinatorial structure of the cosmohedron is very well suited to this general setting. In the Appendix we provide some physical motivation for this general construction, by sketching a new motivating example: we introduce the *one-loop \mathcal{U} -polytope*, which captures the combinatorics of all one-loop cubic (trivalent) graphs, as well as the Feynman polytope for the Symanzik \mathcal{U} -polynomial of each graph, all in one polytope.

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A Appendix: \mathcal{X} in Y polytopes

The main focus of this paper has been to explore and prove the mathematical properties of cosmohedra, which encode the cosmological wavefunction of $\text{Tr}(\phi^3)$ theory. We use this appendix to discuss how the principles that guided the construction of cosmohedra are of much broader relevance.

A.1 The cosmohedron as a guiding example for \mathcal{X} in Y polytopes

As we have learned by now, cosmohedra encode geometrically the combinatorics of maximal Matryoshkas of an n -gon. One way to organize all such Matryoshkas is by dividing them into classes labeled by the underlying collection of triangles – *i.e.* the underlying triangulation. This is what naturally occurs in physics. To write down the cosmological wavefunction, we start by listing all possible cubic (trivalent) graphs, which correspond to all possible triangulations. Then for each cubic graph we consider all possible ways of finding maximal collections of nested subgraphs, which give the possible maximal Matryoshkas compatible with the respective underlying triangulation. The cosmohedron is then the object that puts both combinatorial problems together. In particular, to find the cosmohedron we first needed to find two other polytopes:

- Y. The underlying polytope whose vertices label all possible triangulations/diagrams – which at tree-level is simply the associahedron;
- \mathcal{X} . A class of polytopes, one associated to each triangulation, whose vertices label all possible Matryoshkas compatible with the respective triangulation – which is done by the bracket associahedon X_T for each triangulation T .

Once we have these two objects under control, we “blow up” each vertex v_T of Y into the respective \mathcal{X} polytope X_T , via the chiseling procedure described earlier. At the level of the normal fans, we subdivide each cone of the fan of Y using the corresponding normal fan in the family \mathcal{X} .

Therefore, for any given combinatorial problem, where on top of listing all possible diagrams (Y), we also want to keep track of some further information X_G of **each** diagram G (\mathcal{X}). Once we find the polytopes for \mathcal{X} and Y separately, we can seek to define the polytope which captures the full combinatorics (\mathcal{X} in Y) by the procedure described above.

As it turns out, such situations are ubiquitous in physics, and not just in the mathematical settings discussed in Section 6. In this appendix we will discuss a particular example, simply to illustrate how this construction generalizes to other contexts.

A.2 The \mathcal{U} -polytope for one-loop-integrated amplitudes

At loop-level, the amplitudes are expressed as a sum of *Feynman integrals*, which often diverge in boundaries of the integration domain. Thus it is important to record the asymptotic behavior of the integrands. The most transparent way to do this is by using the projective representation of Feynman integrals in terms of *Schwinger parameters* α_e – one for each edge of the Feynman diagram G – where they take the following form:

$$\mathcal{I}_G = \Gamma(d) \int_{\mathbb{RP}_{>0}^{E-1}} \frac{\prod_{e \in G} d\alpha_e}{\text{GL}(1)} \frac{1}{\mathcal{U}_G^{D/2-d} \mathcal{F}_G^d}.$$

Here $d = E - LD/2$ is the *superficial degree of divergence* of a graph G with E edges and genus L (number of independent loops) in spacetime dimension D , and \mathcal{U}_G and \mathcal{F}_G

are the first and second *Symanzik polynomials* of G :

$$\mathcal{U}_G = \sum_{T \text{ tree of } G} \prod_{e \notin T} \alpha_e, \quad \mathcal{F}_G = \sum_{T \text{ 2-tree of } G} \prod_{e \notin T} \alpha_e.$$

The sum in \mathcal{U}_G is over all spanning trees of G , and the sum in \mathcal{F}_G is over all spanning *2-trees* $T_1 \sqcup T_2$, where T_1 and T_2 are two vertex-disjoint trees that span all vertices of G . To correctly describe the divergences of \mathcal{I}_G , one needs to characterize the asymptotic behavior of \mathcal{U}_G and \mathcal{F}_G .

Therefore, at one-loop level, we are looking for an object that captures not only all possible cubic (trivalent) one-loop graphs G (collected in a polytope Y), but also the possible leading monomials in \mathcal{U}_G and \mathcal{F}_G for each graph G (collected in a family of polytopes \mathcal{X}). As it turns out, the polytopes Y and \mathcal{X} polytopes are known in the physics and mathematics literature [AH, AHHM22, BR25, Edm70]. We will show how to combine them together into a single chiseled polytope. This answers a question posed in [AHFS⁺25].

In this Appendix we illustrate the essential ideas behind the construction in a simple but already interesting case: the asymptotic behaviors of the \mathcal{U} -polynomials in the one-loop setting. In an upcoming paper we will treat the general setting of higher loops, and consider both Symanzik polynomials \mathcal{F} and \mathcal{U} .

A.2.1 The Y polytope: the one-loop associahedron

The *one-loop associahedron* is a polytope designed to parameterize the triangulations of a polygon $\hat{\diamond}_n$ with one puncture. Consider an n -gon with vertices cyclically labeled $1, 2, \dots, n$ and a puncture labeled O . We consider curves of two kinds:

1. *chords* $x_{i,j}$ that go from i to j while keeping the puncture O to the right, for $1 \leq i, j \leq n$. (Note that we have *diagonals* $x_{i,j} \neq x_{j,i}$ for each $i \neq j$, and *tadpoles* $x_{i,i}$ for each i .)
2. *spokes* y_k and \tilde{y}_k that spiral from k into the puncture clockwise and counterclockwise, respectively. (For simplicity we draw straight lines from k to O tagged in blue and red, respectively.)

We are interested in the triangulations of $\hat{\diamond}_n$ using these curves. Notice that every triangulation uses exactly n curves, and contains at least one spoke y_k or \tilde{y}_k . Also notice that a clockwise and a counterclockwise spiral always intersect, so every triangulation is monochromatic. The left panels of Figures 20 and 21 show the 6 triangulations (and six coarser subdivisions) for $n = 2$, and the 10 blue triangulations for $n = 3$, respectively.

The *loop associahedron* was first introduced in [AH]; it is a variant of the cluster polytopes in [AHHST22] that includes the *tadpole* curves, $x_{i,i}$. For the punctured n -gon, we get an n -dimensional simplicial polytope whose vertices correspond to the triangulations of $\hat{\diamond}_n$. Its faces correspond to the partial triangulations or subdivisions of $\hat{\diamond}_n$, and the containment relations of the faces match the reverse inclusion relations of the sets of curves. The left panels of Figures 22 and 23 respectively illustrate the one-loop associahedra for $\hat{\diamond}_2$ and $\hat{\diamond}_3$.

A.2.2 The \mathcal{X} polytopes: Feynman polytopes

The asymptotic behavior of the Symanzik polynomials \mathcal{U}_G and \mathcal{F}_G is fully characterized by their Newton polytopes in $\mathbb{R}^{E(G)}$:

$$U_G = \text{Newt}(\mathcal{U}_G), \quad F_G = \text{Newt}(\mathcal{F}_G)$$

called *Feynman polytopes* and studied in [AHHM22].

Since \mathcal{U}_G and \mathcal{F}_G are squarefree, U_G and F_G are 0-1 polytopes. The exponents of \mathcal{U}_G (resp. \mathcal{F}_G) are the complements of the spanning trees (resp. 2-trees) of graph G , which are the bases of the matroid $M(G)$ (resp. the truncation matroid $\text{Tr } M(G)$). It follows that

$$U_G = P_{M(G)^\perp}, \quad F_G = P_{(\text{Tr } M(G))^\perp}$$

where P_M denotes the matroid (basis) polytope of M and $^\perp$ denotes matroid duality.

In this appendix, for clarity and simplicity, we focus on the first Symanzik polynomials in the one-loop setting. When G only has one cycle, the polynomial \mathcal{U}_G is simply the sum of all variables α_e associated to the edges e of that cycle, and the Feynman polytope is just the standard simplex on those edges. We summarize this by writing that $U_G = \Delta_{\text{Cycle}(G)}$ in the one-loop setting. Note that, unlike in the setting of cosmohedra, this polytope is usually low-dimensional.

A.2.3 Towards a general construction: \mathcal{X} in Y fans in dimensions 2 and 3

To motivate the general construction, let us first carry it out in detail in dimensions 2 and 3, beginning at the level of fans. The fan of the one-loop associahedron has a ray for each curve of $\hat{\diamond}_n$, and a face cone(\mathcal{C}) for each collection of $\mathcal{C} \subset \text{Curves}(\hat{\diamond}_n)$ of compatible curves, or equivalently, each subdivision of $\hat{\diamond}_n$. We need to subdivide the cones of this fan to keep track of the dominant terms of the corresponding \mathcal{U} -polynomials.

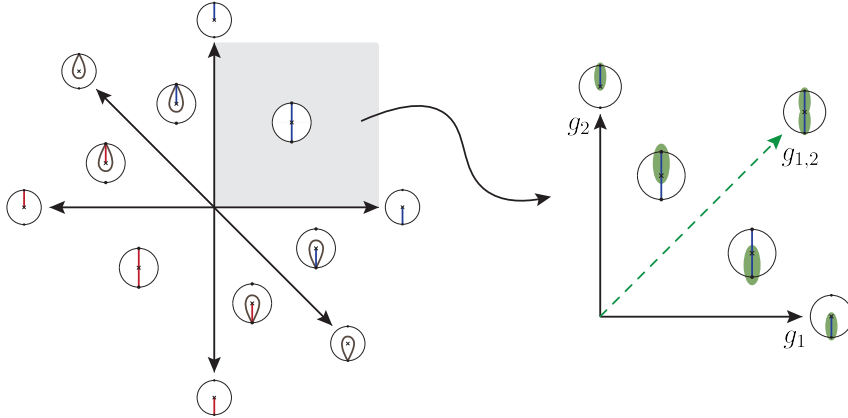


Figure 20: (a) Fan of the two-dimensional one-loop associahedron. (b) Refinement of a cone in the fan to encode the dominating terms of the U -polynomial.

Example A.1. For $n = 2$ there are three blue and three red triangulations of $\hat{\diamond}_2$. Their cones naturally fit into a complete fan in the plane shown in the left panel of Figure 20. The only diagrams with a non-trivial \mathcal{U} polynomial are the ones with two spokes, corresponding to the positive and negative quadrants of the fan. The dual diagram of these triangulations is the bubble diagram with two parallel edges α_1 and α_2 ; its \mathcal{U} -polynomial is simply: $\mathcal{U}\left(\text{---}\begin{array}{c} \alpha_2 \\ \circlearrowleft \\ \alpha_1 \end{array}\text{---}\right) = \alpha_1 + \alpha_2$.

We wish to divide each such cone into two subcones, depending on which term of \mathcal{U} dominates: α_1 or α_2 . We can easily do this by subdividing them in half, adding the rays $g_{1,2} = g_1 + g_2$ and $\tilde{g}_{1,2} = \tilde{g}_1 + \tilde{g}_2$ respectively. Here g_i are the usual g -vectors for the spokes curves, which are the rays of the loop associahedral fan. This subdivision is shown in the right panel of Figure 20.

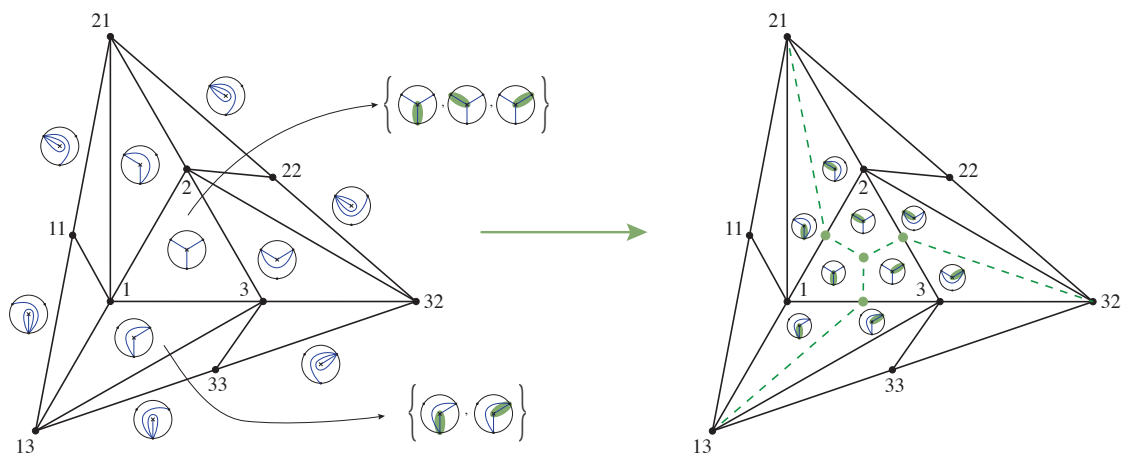


Figure 21: (a) Half-fan of the three-dimensional one-loop associahedron. (b) Half-fan of the three dimensional one-loop \mathcal{U} -polytope.

Example A.2. For $n = 3$ there are 10 blue (and 10 red) triangulations of the punctured disk with three marked points on the boundary $\hat{\diamond}_3$. They label the maximal cones of the fan shown on the left of Figure 21; this is the blue half of the one-loop associahedral fan. The red half is isomorphic, and is glued to this one along the boundary in the opposite orientation. Together they form a complete three-dimensional fan.

Again, for each triangulation \mathcal{C} with two spokes y_i and y_j , we introduce a new ray $g_{ij} = g_i + g_j$ that subdivides the corresponding cone in half. But now there is also a triangulation with three spokes y_1, y_2, y_3 , with \mathcal{U} polynomial $\alpha_1 + \alpha_2 + \alpha_3$. We subdivide its face cone(g_1, g_2, g_3) by introducing four new rays $g_{1,2} = g_1 + g_2$, $g_{2,3} = g_2 + g_3$, $g_{1,3} = g_1 + g_3$, and that span three full-dimensional subcones: cone₁ = cone($g_1, g_{1,2}, g_{1,3}, g_{1,2,3}$), cone₂ = cone($g_2, g_{1,2}, g_{2,3}, g_{1,2,3}$), and cone₃ = cone($g_3, g_{1,3}, g_{2,3}, g_{1,2,3}$). This is illustrated on the right side of Figure 21. Already in this example we see that the fan is not simplicial, and therefore the \mathcal{U} -polytope will not be simple.

A.2.4 The normal fan of the \mathcal{U} -polytope

Now we are ready to describe the general construction of the \mathcal{U} -fan.

Fan of the one-loop associahedron. The fan of the one-loop associahedron [AH] has a ray for each curve of $\widehat{\diamond}_n$, and a face $\text{cone}(\mathcal{C})$ for each collection of compatible curves, $\mathcal{C} \subset \text{Curves}(\widehat{\diamond}_n)$, or equivalently, each subdivision of $\widehat{\diamond}_n$. This is a simplicial fan whose cones are ordered by reverse inclusion:

$$\text{cone}(\mathcal{C}) \subseteq \text{cone}(\mathcal{C}') \quad \text{if and only if} \quad \mathcal{C} \subseteq \mathcal{C}'.$$

Fan of the \mathcal{U} polytope. Now consider a facet $\text{cone}(\mathcal{C})$ of the one-loop associahedron, corresponding to a triangulation \mathcal{C} of $\widehat{\diamond}_n$. In the polytope, we wish to replace the vertex $v_{\mathcal{C}}$ with a simplex $\Delta_{\text{Spokes}(\mathcal{C})}$, since the spokes of \mathcal{C} correspond to the edges in the unique loop of the cubic diagram dual to \mathcal{C} . Therefore we need to subdivide the simplicial face $\text{cone}(\mathcal{C})$ into $|\text{Spokes}(\mathcal{C})|$ maximal cones. If $\text{cone}(\mathcal{C})$ is regular then the s -th cone $\text{cone}(\mathcal{C}, s)$ consists of the points in $\text{cone}(\mathcal{C})$ whose closest spoke-ray is g_s . If it is not, we linearly transform it into a regular cone, and use that subdivision. Explicitly, we obtain new rays

$$g_{\mathcal{S}} = \sum_{s \in \mathcal{S}} g_s, \quad \tilde{g}_{\mathcal{S}} = \sum_{s \in \mathcal{S}} \tilde{g}_s$$

for each subset $\mathcal{S} \subseteq [n]$ and more generally, a new cone

$$\text{cone}(\mathcal{C}, \mathcal{S}) = \text{cone}(\{g_{ij} : x_{ij} \in \text{Chords}(\mathcal{C})\} \cup \{g_T : \mathcal{S} \subseteq T \subseteq \text{Spokes}(\mathcal{C})\})$$

for each non-crossing set of curves \mathcal{C} and each non-empty subset $\emptyset \neq \mathcal{S} \subseteq \text{Spokes}(\mathcal{C})$ – where we allow the exception $\mathcal{S} = \emptyset$ for the sets \mathcal{C} with no spokes.

Note that $\text{cone}(\mathcal{C}, \mathcal{S})$ is a $|\text{Chords}(\mathcal{C})|$ -fold cone over a $(|\text{Spokes}(\mathcal{C})| - |\mathcal{S}|)$ -cube. We have

$$\text{cone}(\mathcal{C}, \mathcal{S}) \subseteq \text{cone}(\mathcal{C}', \mathcal{S}') \quad \text{if and only if} \quad \mathcal{C} \subseteq \mathcal{C}' \text{ and } \mathcal{S} \supseteq \mathcal{S}'.$$

The blue and red fans glue along the “equatorial complex” of (necessarily lower-dimensional) faces $\text{cone}(\mathcal{C}, \emptyset)$ such that \mathcal{C} has no spokes.

A.2.5 The \mathcal{U} -polytope

We now construct a geometric realization of the \mathcal{U} -polytope, given by the appropriate chiseling procedure that creates, at each vertex of the underlying Y -polytope, the respective \mathcal{X} -polytope. We understood above how to subdivide the normal fan of the loop associahedron to get the normal fan of the \mathcal{U} -polytope – and therefore we have its new facet normals $g_{\mathcal{S}}$ and $\tilde{g}_{\mathcal{S}}$ for $\mathcal{S} \subseteq [n]$. However, as we saw for the cosmohedron, it is a subtle matter to determine how deeply we need to chisel in each direction to obtain the correct polytope. This is controlled by the constants $\epsilon_{\mathcal{S}}$ and $\tilde{\epsilon}_{\mathcal{S}}$ appearing in the new facet inequalities $g_{\mathcal{S}} \cdot x \geq \epsilon_{\mathcal{S}}$ and $\tilde{g}_{\mathcal{S}} \cdot x \geq \tilde{\epsilon}_{\mathcal{S}}$.

Given a vector \mathbf{c} of positive parameters c_{rs} , c_r and \tilde{c}_r for $1 \leq r, s \leq n$, the *one-loop associahedron* $L_1(\mathbf{c})$ defined in [AH, BR25] is the n -dimensional polytope

$$L_1(\mathbf{c}) = \left\{ (x, y, \tilde{y}) \in \mathbb{R}^{\text{Curves}(\widehat{\diamond}_{n+2})} : [x_{r,s} \geq 0, y_r \geq 0, \tilde{y}_r \geq 0], \right. \\ \left. \begin{array}{l} x_{r,r} + x_{r+1,r+1} - x_{r+1,r} - y_r - \tilde{y}_{r-1} = c_{r+1,r}, \\ x_{r,s+1} + x_{r-1,s} - x_{r,s} - x_{r-1,s-1} = c_{rs} \text{ for } r \neq \{s+1, s\}, \\ y_r + \tilde{y}_r - x_{r+1,r+1} = c_{r+1}. \end{array} \right\} \quad (13)$$

where we set to 0 any variables that are undefined because their subscripts are outside of the prescribed ranges. In this equation we are writing vectors in $\mathbb{R}^{\text{Curves}(\widehat{\diamond}_{n+2})}$ in the form $(x, y, \tilde{y}) \in \mathbb{R}^{\text{Chords}(\widehat{\diamond}_{n+2})} \times \mathbb{R}^{\text{Spokes}(\widehat{\diamond}_{n+2})} \times \mathbb{R}^{\text{Spokes}(\widehat{\diamond}_{n+2})}$ where $x = (x_{r,s})_{1 \leq r, s \leq n}$, $y = (y_r)_{1 \leq r \leq n}$, and $\tilde{y} = (\tilde{y}_r)_{1 \leq r \leq n}$.

To construct our \mathcal{U} -polytope, we need to introduce new facet inequalities of the form:

$$\sum_{k \in \mathcal{S}} y_k \geq \epsilon_{\mathcal{S}}, \quad \sum_{k \in \mathcal{S}} \tilde{y}_k \geq \tilde{\epsilon}_{\mathcal{S}} \quad \text{for } \emptyset \neq \mathcal{S} \subseteq [n] \quad (14)$$

for the possible subsets of spokes. A first condition on the ϵ s is that they should be small enough in relation to the c s that the chiseling will not remove any faces of $L_1(\mathbf{c})$. The parameters must also satisfy the equalities

$$\epsilon_{\mathcal{S}_1} + \epsilon_{\mathcal{S}_2} = \epsilon_{\mathcal{S}_1 \cup \mathcal{S}_2} + \epsilon_{\mathcal{S}_1 \cap \mathcal{S}_2} \text{ for any subsets } \mathcal{S}_1, \mathcal{S}_2 \text{ with } \mathcal{S}_1 \cap \mathcal{S}_2 \neq \emptyset \quad (15)$$

since the corresponding rays of the normal fan satisfy $g_{\mathcal{S}_1} + g_{\mathcal{S}_2} = g_{\mathcal{S}_1 \cup \mathcal{S}_2} + g_{\mathcal{S}_1 \cap \mathcal{S}_2}$, and they lie in the same cone $\text{cone}([n], \{y_s\})$ of the fan for any $s \in \mathcal{S}_1 \cap \mathcal{S}_2$. In addition, the ϵ s must satisfy the *wall-crossing inequalities*

$$\epsilon_{i,j} > \epsilon_i + \epsilon_j \quad \text{for } i \neq j \quad (16)$$

since the corresponding rays satisfy $g_{i,j} = g_i + g_j$, and $\text{cone}(g_i, g_{i,j})$ and $\text{cone}(g_j, g_{i,j})$ are distinct cones in the fan. The analogous conditions hold for the $\tilde{\epsilon}$'s. One may verify that these are all the necessary and sufficient conditions to obtain the desired \mathcal{U} -polytope.

The solutions to the system of equations (15) are of the form

$$\epsilon_{\{s_1, \dots, s_k\}} = \epsilon_{s_1} + \dots + \epsilon_{s_k} + (k-1)\epsilon \quad (17)$$

for any choice of $\epsilon_1, \dots, \epsilon_n, \epsilon$, and the inequalities (16) are satisfied when $\epsilon > 0$. The same analysis holds for the $\tilde{\epsilon}$'s. Thus a particularly simple choice of parameters that cuts out a \mathcal{U} -polytope is

$$\epsilon_{\mathcal{S}} = \tilde{\epsilon}_{\mathcal{S}} = (|\mathcal{S}| - 1)\epsilon. \quad (18)$$

for small $\epsilon > 0$. Summarizing, the \mathcal{U} -polytope is

$$\mathcal{U}(\mathbf{c}) = \{(x, y, \tilde{y}) \in L_1(\mathbf{c}) : \sum_{k \in \mathcal{S}} y_k \geq (|\mathcal{S}| - 1)\epsilon, \sum_{k \in \mathcal{S}} \tilde{y}_k \geq (|\mathcal{S}| - 1)\epsilon \text{ for } \emptyset \neq \mathcal{S} \subseteq [n]\}$$

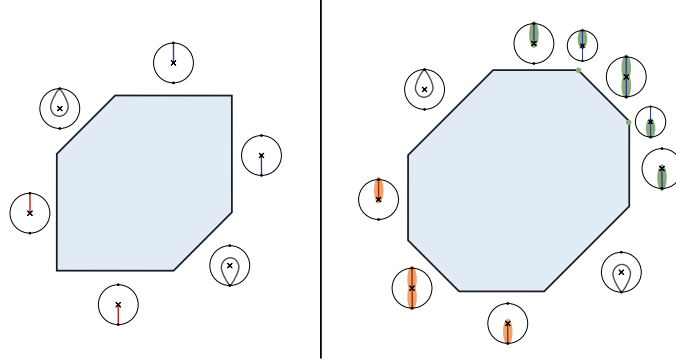


Figure 22: (a) Y : Two-dimensional one-loop associahedron. (b) \mathcal{X} in Y : Two-dimensional one-loop \mathcal{U} -polytope.

Example A.3. For $n = 2$, the one-loop associahedron is the intersection of the subspace

$$\begin{aligned} y_1 + \tilde{y}_1 - x_{2,2} &= c_2, & y_2 + \tilde{y}_2 - x_{1,1} &= c_1, \\ x_{1,1} + x_{2,2} - y_2 - \tilde{y}_1 &= c_{1,2}, & x_{1,1} + x_{2,2} - y_1 - \tilde{y}_2 &= c_{2,1}, \end{aligned} \quad (19)$$

with the positive orthant $x_{1,1} \geq 0, x_{2,2} \geq 0, y_1 \geq 0, y_2 \geq 0$. To obtain the \mathcal{U} -polytope, we impose the stronger inequalities

$$x_{1,1} \geq 0, x_{2,2} \geq 0, y_1 \geq \epsilon_1, y_2 \geq \epsilon_2, y_1 + y_2 \geq \epsilon_1 + \epsilon_2 + \epsilon. \quad (20)$$

for any $\epsilon_1, \epsilon_2, \epsilon > 0$ and the analogous inequalities for \tilde{y} for any $\tilde{\epsilon}_1, \tilde{\epsilon}_2, \tilde{\epsilon}$ with $\epsilon, \tilde{\epsilon} > 0$. Using the parameterization $\epsilon_1 = \epsilon_2 = 0$ (and similarly for $\tilde{\epsilon}$) and $\epsilon = \tilde{\epsilon}$ we obtain the correct embedding as shown on the right of Figure 22.

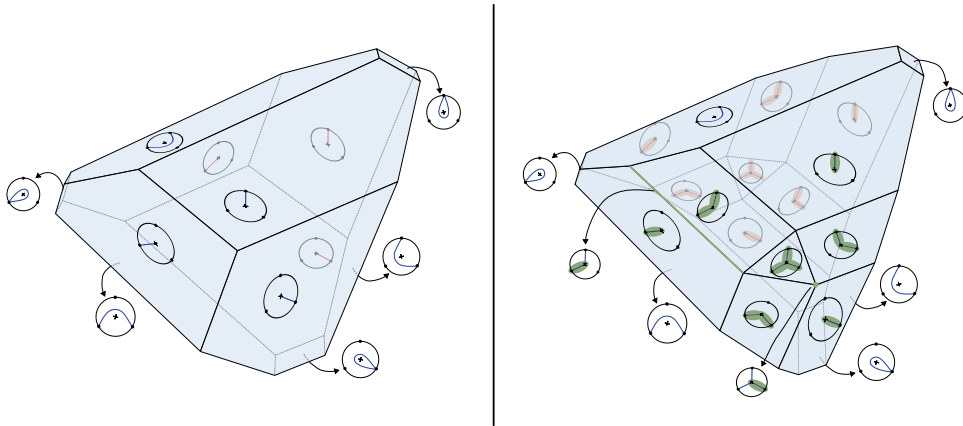


Figure 23: (a) Y : Three-dimensional one-loop associahedron. (b) \mathcal{X} in Y : Three dimensional one-loop \mathcal{U} -polytope.

Similarly, for $n = 3$, we intersect the subspace

$$\begin{aligned}
y_1 + \tilde{y}_1 - x_{2,2} &= c_2, & y_2 + \tilde{y}_2 - x_{3,3} &= c_3, & y_3 + \tilde{y}_3 - x_{1,1} &= c_1, \\
x_{1,1} + x_{2,2} + x_{2,1} - y_1 - \tilde{y}_3 &= c_{2,1}, & x_{2,2} + x_{3,3} + x_{3,2} - y_2 - \tilde{y}_1 &= c_{3,2}, \\
x_{3,3} + x_{1,1} + x_{1,3} - y_3 - \tilde{y}_2 &= c_{1,3}, & x_{1,3} + x_{2,1} - x_{1,1} &= c_{2,3}, \\
x_{2,1} + x_{3,2} - x_{2,2} &= c_{3,1}, & x_{3,2} + x_{1,3} - x_{3,3} &= c_{1,2},
\end{aligned} \tag{21}$$

with the inequalities

$$\begin{aligned}
x_{i,i} &\geq 0, & x_{2,1} &\geq 0, & x_{3,2} &\geq 0, & x_{1,3} &\geq 0, \\
y_i &\geq \epsilon_i, & y_i + y_j &\geq \epsilon_i + \epsilon_j + \epsilon, & y_1 + y_2 + y_3 &\geq \epsilon_1 + \epsilon_2 + \epsilon_3 + 2\epsilon
\end{aligned} \tag{22}$$

for $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon > 0$, and analogous inequalities for the \tilde{y} s.

With $\epsilon_i = \tilde{\epsilon}_i = 0$ and $\epsilon = \tilde{\epsilon} = 1$, we produce the correct embedding seen in Figure 23.

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