The Number of Halving Circles

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1. INTRODUCTION. We say that a set S of $2n+1$ points in the plane is in *general position* if no three of the points are collinear and no four are concyclic. We call a circle *halving* with respect to S if it has three points of S on its circumference, $n-1$ points in its interior, and $n-1$ in its exterior. The goal of this paper is to prove the following surprising fact: *any* set of $2n+1$ points in general position in the plane has *exactly* n^2 halving circles.

Our starting point is the following problem, which appeared in the 1962 Chinese Mathematical Olympiad [**7**].

Problem 1. *Prove that any set of* $2n+1$ *points in general position in the plane has a halving circle.*

For the rest of sections 1 and 2, n is a fixed positive integer and S signifies an arbitrary set of $2n+1$ points in general position in the plane.

There are several solutions to Problem 1. One possible approach is the following. Let A and B be two consecutive vertices of the convex hull of S. We claim that some circle going through A and B is halving. All circles through A and B have their centers on the perpendicular bisector ℓ of the segment AB. Pick a point O on ℓ that lies on the same side of AB as S and is sufficiently far away from AB that the circle Γ with center O and passing through A and B completely contains S. This can clearly be done. Now slowly "push" O along ℓ, moving it towards AB. The circle Γ changes continuously with O. As we do this, Γ stops containing some points of S. In fact, it loses the points of S one at a time: if it lost P and Q simultaneously, then points P, Q, A, and B would be concyclic. We can move O sufficiently far away past AB that, in the end, the circle does not contain any points of S.

Originally, Γ contained all the points of S. Now, as it loses one point of S at a time in this process, we can decide how many points we want it to contain. In particular, if we stop moving O when the circle is about to lose the nth point P of S, then the resulting Γ is halving: it has A, B, and P on its circumference, $n-1$ points inside it, and $n-1$ outside it, as illustrated in Figure 1. ∎

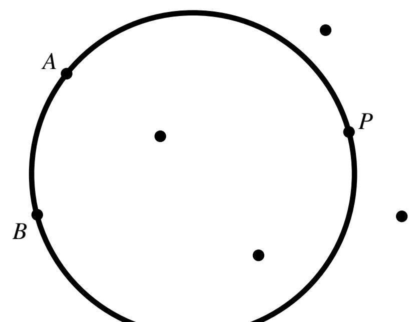

Figure 1. A halving circle through A, B, and P.

The foregoing proof shows that any set S has several different halving circles. We can certainly construct one for each pair of consecutive vertices of the convex hull of S. In fact, the argument can be modified to show that, for *any* two points of S, we can find a halving circle passing through them.

This suggests that we ask the following question: What can we say about the number N_S of halving circles of S? At first sight, it seems that we really cannot say very much at all about this number. Halving circles seem hard to "control," and harder to count.

We should, however, be able to find upper and lower bounds for N_S in terms of n. From the start we know that $N_S \geq n(2n+1)/3$, since we can find a halving circle for each pair of points of S, and each such circle is counted by three different pairs. Computing an upper bound seems more difficult. If we fix points A and B of S, it is indeed possible that all $2n-1$ circles through A, B, and some other point of S are halving. The reader is invited to check this. Such a situation is not likely to arise very often for a set S. However, it is not clear how to make this idea precise, and then use it to obtain a nontrivial upper bound.

For $n = 2$, it is not too difficult to check by hand that $N_S = 4$ for *any* set S of five points in general position in the plane. This result first appeared in [**6**]. It was also proposed, but not chosen, as a problem for the 1999 International Mathematical Olympiad. Notice that our lower bound gives $N_S \geq 4$.

In a different direction, a problem of the 1998 Asian-Pacific Mathematical Olympiad, proposed by the author, asserted the following:

Problem 2. *N_S has the same parity as n.*

Problem 2 follows easily from the nontrivial observation that, for any A and B in S, the number of halving circles that go through A and B is odd. We leave the proof of this observation as a nice exercise.

Amazingly, it turns out that we can say something much stronger. The following result supercedes the previous considerations.

Theorem 1. *Any set of $2n+1$ points in general position in the plane has exactly n^2 halving circles.*

Theorem 1 is the main result of this paper. In section 2 we prove that every set of $2n+1$ points in general position in the plane has the same number of halving circles. In section 3 we prove that this number is exactly n^2, and we present a generalization.

2. THE NUMBER OF HALVING CIRCLES IS CONSTANT. At this point, we could cut to the chase and prove the very counterintuitive Theorem 1. At the risk of making the argument seem slightly longer, we believe that it is worthwhile to present the motivation behind its discovery. Therefore, we ask the reader to forget momentarily the punchline of this article.

Suppose that we are trying to find out whatever we can about the number N_S of halving circles of S. As mentioned in the introduction, this number does not seem very tractable and it is not clear how much we can say about it. Being optimistic, we might hope to be able to answer the following two questions.

Question 1. *What are the sharp lower and upper bounds $m = m_{2n+1}$ and $M = M_{2n+1}$ for N_S?*

Question 2. *What are all the values that N_S takes in the interval $[m, M]$?*

Question 1 would appear to present considerable difficulty. To answer it completely, we would first need to prove an inequality $m \leq N_S \leq M$, and then construct suitable sets $S_{\min}$ and $S_{\max}$ that achieve these bounds. Let us focus on Question 2 instead. Here is a first approach.

Suppose that we start with the set $S_{\min}$ (with $N_S = m$) and move its points continuously so as to end up with $S_{\max}$ (with $N_S = M$). We might guess that the value of N_S should change "continuously," in the sense that N_S should sweep out all the integers between m and M as S moves from a minimal to a maximal configuration.

We know immediately that this would be overly optimistic. From Problem 2 we learn that the parity of N_S is determined by n, so N_S does not assume *all* integral values between m and M. In any case, the natural question to ask is: What kind of changes does the value of N_S undergo as S changes continuously?

Let

$$S_{\min} = \{P_1, \ldots, P_{2n+1}\}, \qquad S_{\max} = \{Q_1, \ldots, Q_{2n+1}\}.$$

Now slowly transform $S_{\min}$ into $S_{\max}$: first send P_1 to Q_1 continously along some path, then send P_2 to Q_2 continuously along some other path, and so on. We can think of our set S as changing with time. At the initial time $t = 0$, our set is $S(0) = S_{\min}$. At the final time $t = T$, our set is $S(T) = S_{\max}$. In between, $S(t)$ varies continously with respect to t. Must $N_{S(t+\Delta t)} - N_{S(t)}$ be small when Δt is small? (As we move from $S(0)$ to $S(T)$ continuously, it is likely that several intermediate sets $S(t)$, with $0 < t < T$, are not in general position. We shall see that we can go from $S(0)$ to $S(T)$ in such a manner that we encounter only finitely many such sets. When $S(t)$ is not in general position, we still need to know whether $N_{S(t+\Delta t)} - N_{S(t-\Delta t)}$ must be small when Δt is small.)

In the way we defined the deformation from $S_{\min}$ to $S_{\max}$, the points of S move one at a time. Let us focus for the moment on the interval of time during which P_1 moves towards Q_1.

Suppose that the number N_S changes between time t and time $t + \Delta t$. Then it must be the case that for some i, j, k, and l the circle $P_i P_j P_k$ surrounds (or does not surround) point P_l at time t, but at time $t + \Delta t$ it does not (or does) encircle P_l. For this to be true, it must happen that, sometime between t and $t + \Delta t$, either these four points are concyclic or three of them are collinear. Since P_1 is the only point that moves in this process, we can conclude that P_1 must cross a circle or a line determined by the other points; this is what causes N_S to change. We will call the circles and lines determined by the points $P_2, P_3, \ldots, P_{2n+1}$ the *boundaries*.

We are free to choose the path along which P_1 moves towards Q_1. To make things easier, we may assume that P_1 never crosses two of the boundaries at the same time. This can clearly be guaranteed: we know that these boundaries intersect pairwise in finitely many points, and all we have to do is avoid their intersection points in the path from P_1 to Q_1. We can also assume that Δt is small enough that P_1 crosses exactly one boundary between times t and $t + \Delta t$. Let us see how N_S can change in this time interval.

It will be convenient to call a circle $P_i P_j P_k$ (a, b)*-splitting* (where $a + b = 2n - 2$) if it has a points of S inside it and the remaining b points outside it. The halving circles are the $(n-1, n-1)$-splitting circles.

Assume first that P_1 crosses line $P_i P_j$ in going from position $P_1(t) = A$ to position $P_1(t + \Delta t) = B$. From the remarks made earlier, we know that only circle $P_1 P_i P_j$ can change the value of N_S by becoming or ceasing to be halving. Assume that circle $A P_i P_j$ is (a, b)-splitting. Since P_1 only crosses the boundary $P_i P_j$ when going from A

to B, the region common to circles AP_iP_j and BP_iP_j cannot contain any points of S, as indicated in Figure 2. The region outside of both circles cannot contain points of S either. For circle AP_iP_j to be (a, b)-splitting, the other two regions must then contain a and b points, respectively, as shown. Therefore circle BP_iP_j is (b, a)-splitting. It follows that AP_iP_j is halving if and only if BP_iP_j is halving (if and only if $a = b = n - 1$). Somewhat surprisingly, we conclude that the value of N_S does not change when P_1 crosses a line determined by the other points; it can only change when P_1 crosses a circle.

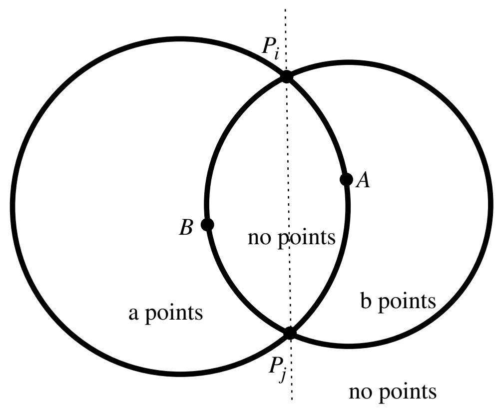

Figure 2. P_1 crosses line P_iP_j.

Now assume that P_1 crosses circle $P_iP_jP_k$ in moving from position $P_1(t) = A$ inside the circle to position $P_1(t + \Delta t) = B$ outside it, as shown in Figure 3. (The other case, when P_1 moves into the circle, is analogous.) The value of N_S can change only by circles $P_iP_jP_k$, $P_1P_jP_k$, $P_1P_kP_i$, and $P_1P_iP_j$ becoming or ceasing to be halving. We can assume that P_1 crosses the arc P_iP_j of the circle that does not contain point P_k. Notice that A must be outside triangle $P_iP_jP_k$ if we want P_1 to cross only one boundary in the time interval considered. Assume that circle $P_iP_jP_k$ is (a, b)-splitting when $P_1 = A$. As before, we know that the only regions of Figure 3 containing points of S are the one common to circles AP_iP_j and BP_iP_j and the one outside both of them.

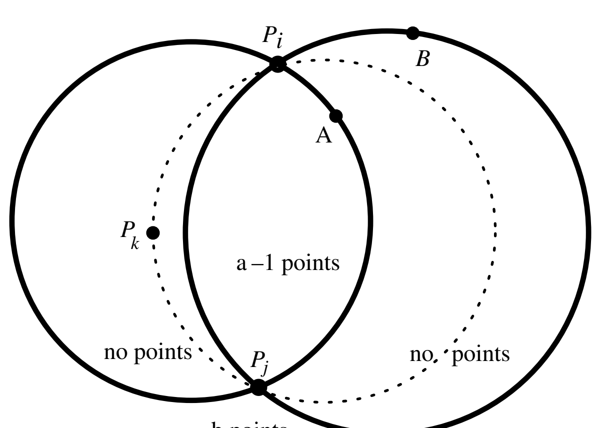

Figure 3. P_1 crosses circle $P_iP_jP_k$.

They must contain $a-1$ and b points, respectively. Circle $P_iP_jP_k$ goes from being (a,b)-splitting to being $(a-1,b+1)$-splitting. The same is true of circle $P_1P_iP_j$.

It is also not hard to see, by a similar argument, that circles $P_1P_jP_k$ and $P_1P_kP_i$ both change from being $(a-1,b+1)$-splitting to being (a,b)-splitting. Again, the key assumption is that P_1 crosses only the boundary $P_iP_jP_k$ in this time interval.

So, by having P_1 cross circle $P_iP_jP_k$, we have traded two (a,b)-splitting and two $(a-1,b+1)$-splitting circles for two $(a-1,b+1)$-splitting and two (a,b)-splitting circles, respectively. It follows that the number N_S of halving circles also remains constant when P_1 crosses a circle $P_iP_jP_k$!

We had shown that, as we moved P_1 to Q_1, N_S could only possibly change in a time interval when P_1 crossed a boundary determined by the other points. But now we see that, even in such a time interval, N_S does not change! Therefore moving P_1 to Q_1 does not change the value of N_S. Similarly, moving P_i to Q_i does not change N_S for $i=2,3,\dots,2n+1$. It follows that N_S is the same for $S_{\min}$ and $S_{\max}$. In fact, N_S is the same for any set S of $2n+1$ points in general position!

3. THE NUMBER OF HALVING CIRCLES IS n^2. Now that we know that the number N_S depends only on the number of points in S, let N_{2n+1} be the number of halving circles for a set of $2n+1$ points in general position. We compute N_{2n+1} recursively.

Construct a set S of $2n+1$ points as follows. First consider the vertices of a regular $(2n-1)$-gon with center O. Now move them very slightly so that they are in general position. Label them $P_1,\dots,P_{2n-1}$ clockwise. The deformation should be sufficiently slight that all the lines OP_i still split the remaining points into two sets of equal size, and all the circles $P_iP_jP_k$ still contain O. Also consider a point Q located sufficiently far away from the others that it lies outside all the circles formed by the points considered so far. Of course, we need Q to be in general position with respect to the remaining points. We count the number of halving circles of $S=\{O,P_1,\dots,P_{2n-1},Q\}$.

First consider the circles of the form $P_iP_jP_k$. These circles contain O and do not contain Q, so they are halving for S if and only if they are halving for $\{P_1,\dots,P_{2n-1}\}$. Thus there are N_{2n-1} such circles.

Next consider the circles OP_iP_j. It is clear that these circles contain at most $n-2$ other P_ks. They do not contain Q, so they contain at most $n-2$ points, and they are not halving.

Finally consider the circles that go through Q and two other points X and Y of S. Circle QXY splits the remaining points in the same way that line XY does. More precisely, circle QXY contains a point P of S if and only if P is on the same side of line XY that Q is. This follows easily from the fact that Q lies outside circle PXY. Therefore we have to determine which lines determined by two points of $S-\{Q\}$ split the remaining points of this set into two subsets of $n-1$ points each. This question is much easier to answer: the lines OP_i do this and the lines P_iP_j do not. It follows that the $2n-1$ circles OP_iQ are halving, and the circles P_iP_jQ are not.

To summarize: the halving circles of S are the N_{2n-1} halving circles of $\{P_1,\dots,P_{2n-1}\}$ and the $2n-1$ circles OP_iQ. Therefore $N_{2n+1}=N_{2n-1}+2n-1$. Since $N_3=1$, it follows inductively that $N_{2n+1}=n^2$. This completes the proof of Theorem 1.

Theorem 2. *Consider a set of $2n+1$ points in general position in the plane, and two nonnegative integers a and b satisfying $a<b$ and $a+b=2n-2$. There are exactly $2(a+1)(b+1)$ circles that are either (a,b)-splitting or (b,a)-splitting.*

Sketch of proof. The argument of section 2 carries over directly to this situation and shows that the number of circles under consideration, which we denote $N(a, b)$, depends only on a and b. Therefore, it suffices to compute it for the set S constructed in the proof of Theorem 1.

Just as earlier, there are $N(a-1, b-1)$ such circles among the circles $P_i P_j P_k$. Among the $O P_i P_j$ there are exactly $2n-1$ such circles, namely, the circles $O P_i P_{i+a+1}$ (taking subscripts modulo $2n-1$). There are also $2n-1$ such circles among the $Q P_i P_j$, namely, the circles $Q P_i P_{i+a+1}$. Finally, there are no such circles among the $O P_i Q$. Therefore

$$N(a, b) = N(a-1, b-1) + 4n - 2 = N(a-1, b-1) + 2a + 2b + 2.$$

For $a = 0$, we get that $N(0, b) = 2b + 2$. Theorem 2 then follows by induction. ■

It is worth mentioning that our study is closely related to the Voronoi diagram and the Delaunay triangulation of a point configuration. The language of oriented matroids provides a very nice explanation of this connection (for details, see [**1**, sec. 1.8]). In fact, Theorems 1 and 2 are essentially equivalent to a beautiful result of D. T. Lee [**4**], which gives a sharp bound for the number of vertices of an order j Voronoi diagram. See [**2**, Theorem 3.5] for another proof.

Under a stereographic projection, the halving circles of a point configuration in the plane correspond to the halving planes of a point configuration on a sphere in three-dimensional space. More generally, we could also attempt to count the halving hyperplanes of a point configuration in n-dimensional space. This problem belongs to the vast literature on k-sets and j-facets, where exact enumerative results are very rare. As an introduction, we recommend [**5**, chap. 11] to the interested reader.

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