The equivariant Ehrhart theory of the permutahedron

Federico Ardila* Mariel Supina† Andrés R. Vindas-Meléndez‡

Abstract

Equivariant Ehrhart theory enumerates the lattice points in a polytope with respect to a group action. Answering a question of Stapledon, we describe the equivariant Ehrhart theory of the permutahedron, and we prove his Effectiveness Conjecture in this special case.

1 Introduction

Ehrhart theory measures a polytope P by counting the lattice points in its dilations tP for $t \in \mathbb{N}$. Stapledon [11] introduced *equivariant Ehrhart theory* as a refinement of Ehrhart theory that takes into account the symmetries of the polytope P. He asked for a description of the equivariant Ehrhart theory of the permutahedron under its group of symmetries, the symmetric group. This study was initiated in [2], which computed the equivariant volumes of the permutahedron. In this paper we complete the answer to Stapledon's question, computing the equivariant Ehrhart polynomials of the permutahedron, and verifying several conjectures in this special case.

1.1 The Ehrhart quasipolynomials of the fixed polytopes of the permutahedron

We consider the action of the symmetric group S_n on the $(n-1)$-dimensional permutahedron Π_n. For each permutation $\sigma \in S_n$, we define the *fixed polytope* $\Pi_n^\sigma \subseteq \Pi_n$ to be the subset of the permutahedron Π_n fixed by σ. Our first main result is a combinatorial formula for its lattice-point enumerator $L_{\Pi_n^\sigma}(t) := |t\Pi_n^\sigma \cap \mathbb{Z}^n|$:

Theorem 1.1. *Let σ be a permutation of $[n]$ and let $\lambda = (\ell_1, \ldots, \ell_m)$ be the partition of $[n]$ given by the lengths of the cycles of σ. Say a set partition $\pi = \{B_1, \ldots, B_k\}$ of $[m]$ is* λ-compatible *if for each block B_i, either ℓ_j is odd for some $j \in B_i$, or the minimum 2-valuation among $\{\ell_j : j \in B_i\}$ is attained at least twice. Also write*

$$v_\pi = \prod_{i=1}^{k} \left(\gcd_{j \in B_i} \ell_j \cdot \Big(\sum_{j \in B_i} \ell_j \Big)^{|B_i|-2} \right). \tag{1}$$

Then the Ehrhart quasipolynomial of the fixed polytope of the permutahedron Π_n fixed by σ is

$$L_{\Pi_n^\sigma}(t) = \begin{cases} \displaystyle\sum_{\pi \models [m]} v_\pi \cdot t^{m-|\pi|} & \text{if } t \text{ is even} \\ \displaystyle\sum_{\substack{\pi \models [m] \\ \lambda-compatible}} v_\pi \cdot t^{m-|\pi|} & \text{if } t \text{ is odd} \end{cases}.$$

*San Francisco State University; U. de Los Andes; federico@sfsu.edu.
†University of California, Berkeley; mariel_supina@berkeley.edu.
‡San Francisco State University; University of Kentucky; andres.vindas@uky.edu.
The authors were supported by NSF Award DMS-1600609 and DMS-1855610 and a Simons Fellowship (FA), the National Physical Science Consortium (MS), and NSF Graduate Research Fellowship DGE-1247392 (ARVM).

1.2 Equivariant Ehrhart theory

Theorem 1.1 fits into the framework of equivariant Ehrhart theory, as we now explain.

Let G be a finite group acting on $\mathbb{Z}^n$ and $P \subseteq \mathbb{R}^n$ be a d-dimensional lattice polytope that is invariant under the action of G. Let M be the sublattice of $\mathbb{Z}^n$ obtained by translating the affine span of P shifted to the origin, and consider the induced representation $\rho : G \to GL(M)$. We then obtain a family of permutation representations by looking at how ρ permutes the lattice points inside the dilations of P. Let $\chi_{tP} : G \to \mathbb{C}$ denote the permutation character associated to the action of G on the lattice points in the t^{th} dilate of P. We have

$$\chi_{tP}(g) = L_{P^g}(t)$$

where P^g is the polytope of points in P fixed by g and $L_{P^g}(t)$ is its lattice point enumerator.

The permutation characters χ_{tP} live in the ring $R(G)$ of virtual characters of G, which are the integer combinations of the irreducible characters of G. The positive integer combinations are called *effective*; they are the characters of representations of G.

Stapledon encoded the characters χ_{tP} in a power series $H^*[z] \in R(G)[[z]]$ given by

$$\sum_{t \geq 0} \chi_{tP}(g) z^t = \frac{H^*[z](g)}{(1-z)\det(I - g \cdot z)}. \tag{2}$$

We say that $H^*[z] =: \sum_{i \geq 0} H_i^* z^i$ is *effective* if each virtual character H_i^* is a character. Stapledon denoted this series $\varphi[t]$, but we denote it $H^*[z]$ and call it the *equivariant H^*-series* because for the identity element, $H^*[z](e) = h^*[z]$ is the well-studied h^*-polynomial of P.

The main open problem in equivariant Ehrhart theory is to characterize when $H^*[z]$ is effective, and Stapledon offered the following conjecture.

Conjecture 1.2 ([11, Effectiveness Conjecture 12.1])**.** *Let P be a lattice polytope fixed by the action of a group G. The following conditions are equivalent.*

(i) The toric variety of P admits a G-invariant non-degenerate hypersurface.

(ii) The equivariant H^*-series of P is effective.

(iii) The equivariant H^*-series of P is a polynomial.

Our second main result is the following.

Theorem 1.3. *Stapledon's Effectiveness Conjecture holds for the permutahedron under the action of the symmetric group.*

Finally, in Propositions **??**, **??**, and **??** we verify the three remaining conjectures of Stapledon in this case. There is some suprisingly subtle number theory at play in most of our results, as the following examples illustrate.

1.3 Examples

Example 1.4. Let us illustrate these results for the permutahedron Π_4 and the permutation $\sigma = (12)(3)(4)$ which has cycle type $\lambda = (2,1,1)$, illustrated in Figure 1. The fixed polytope $\Pi_4^{(12)}$ is a half-integral hexagon, and one may verify manually that

$$L_{\Pi_4^{(12)}}(t) = \begin{cases} 4t^2 + 3t + 1 & \text{if } t \text{ is even} \\ 4t^2 + 2t & \text{if } t \text{ is odd,} \end{cases} \qquad H^*[z](12) = 1 + 4z + 11z^2 - 2z^3 + \frac{4z^4}{1+z}.$$

Since the H^*-series of Π_4 is not polynomial when evaluated at (12), Stapledon's Conjecture 1.2 predicts that it is also not effective, and that the permutahedral variety X_{Π_4} does not admit an S_4-invariant non-degenerate hypersurface. We verify this in Section 5.

The equivariant Ehrhart quasipolynomials and H^*-series of Π_3 and Π_4 are shown in Tables 2 and 3.

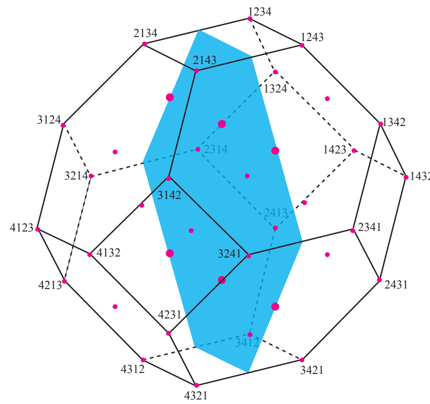

Figure 1: The fixed polytope $\Pi_4^{(12)}$ is a half-integral hexagon containing 6 lattice points.

Example 1.5. Further subtleties already arise in the simple case when Π_n^σ is a segment; this happens when σ has only two cycles of lengths ℓ_1 and ℓ_2. For even t, we simply have

$$L_{\Pi_n^\sigma}(t) = \gcd(\ell_1, \ell_2)t + 1.$$

However, for odd t we have

$$L_{\Pi_n^\sigma}(t) = \begin{cases} \gcd(\ell_1, \ell_2)t + 1 & \text{if } \ell_1 \text{ and } \ell_2 \text{ are both odd,} \\ \gcd(\ell_1, \ell_2)t & \text{if } \ell_1 \text{ and } \ell_2 \text{ have different parity,} \\ \gcd(\ell_1, \ell_2)t & \text{if } \ell_1 \text{ and } \ell_2 \text{ are both even and they have the same 2-valuation,} \\ 0 & \text{if } \ell_1 \text{ and } \ell_2 \text{ are both even and they have different 2-valuations,} \end{cases}$$

We invite the reader to verify that this formula follows from Theorem 1.1.

1.4 Organization

In Section 2 we introduce some background on Ehrhart theory and zonotopes. In Section 3 we compute the Ehrhart polynomial of the fixed polytope Π_n^σ when it is a lattice polytope, and in Section 4 we compute its Ehrhart quasipolynomial in general, proving Theorem 1.1. In Section 5 we compute the equivariant H^*-series $H^*[z]$ for permutahedra and we verify Stapledon's four conjectures on equivariant Ehrhart theory in this special case; most importantly, his Effectiveness Conjecture (Theorem 1.3).

2 Preliminaries

2.1 Ehrhart quasipolynomials

Let P be a convex polytope in $\mathbb{R}^n$. We say that P is a *rational polytope* if all of its vertices are in $\mathbb{Q}^n$. The *lattice point enumerator* of P is the function $L_P : \mathbb{N} \to \mathbb{N}$ given by

$$L_P(t) := |tP \cap \mathbb{Z}^n|;$$

that is, $L_P(t)$ is the number of integer points in the t^{th} dilate of P. A function $f : \mathbb{Z} \to \mathbb{R}$ is *quasipolynomial* if there exists a period d and polynomials $f_0, f_1, \ldots, f_{d-1}$ such that $f(n) = f_i(n)$ whenever $n \equiv i \pmod{d}$.

Theorem 2.1 (Ehrhart's Theorem [4])**.** *If P is a rational polytope, then $L_P(t) := |tP \cap \mathbb{Z}^n|$ is a quasipolynomial in t. Its degree is $\dim P$ and its period divides the least common multiple of the denominators of the coordinates of the vertices of P.*

2.2 Zonotopes

Let V be a finite set of vectors in $\mathbb{R}^n$. The *zonotope* generated by V, denoted $Z(V)$, is defined to be the Minkowski sum of the line segments connecting the origin to $\mathbf{v}$ for each $\mathbf{v} \in V$. We will also adapt the same notation to refer to any translation of $Z(V)$, that is, the Minkowski sum of any collection of line segments whose direction vectors are the elements of V. Zonotopes have a combinatorial decomposition that is useful when calculating volumes and counting lattice points. The following result is due to Shephard.

Proposition 2.2 ([9, Theorem 54]). *A zonotope $Z(V)$ can be subdivided into half-open parallelotopes that are in bijection with the linearly independent subsets of V.*

A linearly independent subset $S \subseteq V$ corresponds under this bijection to the half-open parallelotope

$$\square S := \sum_{\mathbf{v} \in S} (\mathbf{0}, \mathbf{v}].$$

Stanley gave a combinatorial formula for the Ehrhart polynomial of a lattice zonotope.

Theorem 2.3 ([10, Theorem 2.2]). *Let $Z(V)$ be a lattice zonotope generated by V. Then*

$$L_{Z(V)}(t) = \sum_{\substack{S \subseteq V \\ \textit{lin. indep.}}} \operatorname{vol}(\square S) \cdot t^{|S|}. \tag{3}$$

In the statement above and throughout the paper, volumes are normalized so that any primitive lattice parallelotope has volume 1.

2.3 Fixed polytopes of the permutahedron

The symmetric group S_n acts on $\mathbb{R}^n$ by permuting coordinates of points. The *permutahedron* Π_n is the convex hull of the S_n-orbit of the point $(1, 2, \ldots, n) \in \mathbb{R}^n$; that is, of the $n!$ permutations of $[n]$.

Let $\sigma \in S_n$ be a permutation with cycles $\sigma_1, \ldots, \sigma_m$; their lengths form a partition $\lambda = (\ell_1, \ldots, \ell_m)$ of n. For each cycle σ_k of σ, let $\mathbf{e}_{\sigma_k} = \sum_{i \in \sigma_k} \mathbf{e}_i$. The *fixed polytope* Π_n^σ is defined to be the polytope consisting of all points in Π_n that are fixed under the action of σ. We will use a few results from [2], which we now summarize.

Theorem 2.4 ([2, Theorems 1.2 and 2.12]). *The fixed polytope Π_n^σ has the following zonotope description:*

$$\Pi_n^\sigma = \sum_{1 \le i < j \le m} [\ell_i \mathbf{e}_{\sigma_j}, \ell_j \mathbf{e}_{\sigma_i}] + \sum_{k=1}^{m} \frac{\ell_k + 1}{2} \mathbf{e}_{\sigma_k}. \tag{4}$$

Its normalized volume is

$$\operatorname{vol} \Pi_n^\sigma = n^{m-2} \gcd(\ell_1, \ldots, \ell_m). \tag{5}$$

Corollary 2.5. *The fixed polytope Π_n^σ is integral or half-integral. It is a lattice polytope if and only if all cycles of σ have odd length.*

Proof. From (4) and from the fact that all of the $\mathbf{e}_{\sigma_i}$ in (4) are linearly independent, we can see that all the vertices of Π_n^σ will be in the integer lattice if and only if $\ell_i + 1$ is even for all i. $\square$

Equation (4) also shows that Π_n^σ is a rational translation of the zonotope $Z(V)$ where

$$V = \{\ell_i \mathbf{e}_{\sigma_j} - \ell_j \mathbf{e}_{\sigma_i} : 1 \le i < j \le m\}.$$

The following result characterizes the linearly independent subsets of V.

Lemma 2.6 ([2, Lemma 3.2]). *The linearly independent subsets of V are in bijection with forests with vertex set $[m]$, where the vector $\ell_i e_{\sigma_j} - \ell_j e_{\sigma_i}$ corresponds to the edge connecting vertices i and j.*

In light of this lemma, the fixed polytope Π_n^σ gets subdivided into half-open parallelotopes $\square_F$ of the form

$$\square_F = \sum_{\{i,j\}\in E(F)} [\ell_i \mathbf{e}_{\sigma_j}, \ell_j \mathbf{e}_{\sigma_i}] + \sum_{k=1}^{m} \frac{\ell_k+1}{2}\mathbf{e}_{\sigma_k} + v_F, \qquad v_F \in \mathbb{Z}^n \tag{6}$$

for each forest F with vertex set $[m]$. When F is a tree T we have that $\mathrm{vol}(\square_T) = \left(\prod_{i=1}^m \ell_i^{\deg_T(i)-1}\right)\gcd(\ell_1,\ldots,\ell_m)$. by [2, Lemma 3.3]. For a general forest F, the parallelotopes $\square_T$ corresponding to each connected component T of F live in orthogonal subspaces, so

$$\mathrm{vol}(\square_F) = \left(\prod_{j=1}^{m} \ell_j^{\deg_F(j)-1}\right)\left(\prod_{\substack{\text{conn. comp.}\\ T \text{ of } F}} \gcd(\ell_j : j \in \mathrm{vert}(T))\right). \tag{7}$$

3 The Ehrhart polynomial of the fixed polytope: the lattice case

Suppose that $\lambda = (\ell_1, \ldots, \ell_m)$ is a partition of n into odd parts and that $\sigma \in S_n$ has cycle type λ. Then Corollary 2.5 says that Π_n^σ is a lattice zonotope, and hence we can use (3) to write a combinatorial expression for its Ehrhart polynomial. Recall the definition of v_π in (1).

Theorem 3.1. *Let $\sigma \in S_n$ have cycle type $\lambda = (\ell_1, \ldots, \ell_m)$, where ℓ_i is odd for all i. Then*

$$L_{\Pi_n^\sigma}(t) = \sum_{\pi \vDash [m]} v_\pi \cdot t^{m-|\pi|}$$

summing over all partitions $\pi = \{B_1, \ldots, B_k\}$ of $[m]$.

Proof. Combining Theorem 2.3 with (7) gives us the following formula for the Ehrhart polynomial of Π_n^σ:

$$L_{\Pi_n^\sigma}(t) = \sum_{\substack{\text{Forests } F\\ \text{on } [m]}} \left(\prod_{j=1}^{m} \ell_j^{\deg_F(j)-1}\right) \cdot \left(\prod_{\substack{\text{conn. comp.}\\ T \text{ of } F}} \gcd(\ell_j : j \in \mathrm{vert}(T))\right) t^{|E(F)|}. \tag{8}$$

Note that we can construct a forest with vertex set $[m]$ by first partitioning $[m]$ into nonempty sets $\{B_1, \ldots, B_k\}$ and then choosing a tree with vertex set B_j for each j. The number of edges in such a forest is $m-k$. Using these observations, we can rewrite (8) as

$$L_{\Pi_n^\sigma}(t) = \sum_{\{B_1,\ldots,B_k\}\vDash[m]} \left(\prod_{i=1}^{k} \gcd(\ell_j : j \in B_i)\right) \cdot \left(\sum_{\substack{\text{Forests } F\\ \text{inducing}\\ \{B_1,\ldots,B_k\}}} \prod_{j=1}^{m} \ell_j^{\deg_F(j)-1}\right) t^{m-k}.$$

To complete the proof, it remains to show that for a given partition $\pi = \{B_1, \ldots, B_k\}$ of $[m]$, the following identity holds:

$$\sum_{\substack{\text{Forests } F\\ \text{inducing}\\ \{B_1,\ldots,B_k\}}} \prod_{j=1}^{m} \ell_j^{\deg_F(j)-1} = \prod_{i=1}^{k} \left(\sum_{j\in B_i} \ell_j\right)^{|B_i|-2}. \tag{9}$$

This follows from the following identity, found in [2, Lemma 3.4].

$$\sum_{T \text{ tree on } [m]} \prod_{i=1}^{m} x_j^{\deg_T(j)-1} = (x_1 + \cdots + x_m)^{m-2}. \tag{10}$$

Using (10) we obtain

$$\begin{aligned}\sum_{\substack{\text{Forests } F\\ \text{inducing}\\ \{B_1,\dots,B_k\}}} \prod_{j=1}^{m} \ell_j^{\deg_F(j)-1} &= \sum_{\substack{\text{Forests } F\\ \text{inducing}\\ \{B_1,\dots,B_k\}}} \prod_{i=1}^{k} \prod_{j\in B_i} \ell_j^{\deg_F(j)-1}\\ &= \prod_{i=1}^{k} \Bigg(\sum_{\substack{\text{trees } T\\ \text{on } B_i}} \prod_{j\in B_i} \ell_j^{\deg_T(j)-1} \Bigg)\\ &= \prod_{i=1}^{k} \Big(\sum_{j\in B_i} \ell_j \Big)^{|B_i|-2}\end{aligned}$$

as desired. □

4 The Ehrhart quasipolynomial of the fixed polytope: the general case

In general, Π_n^σ is a half-integral polytope. This means that instead of an Ehrhart polynomial, it has an Ehrhart quasipolynomial with period at most 2. As in the lattice case from Section 3, we can decompose Π_n^σ into half-open parallelotopes. However, there is a new feature that does not arise in the lattice case: some of the parallelotopes in this decomposition may not contain any lattice points.

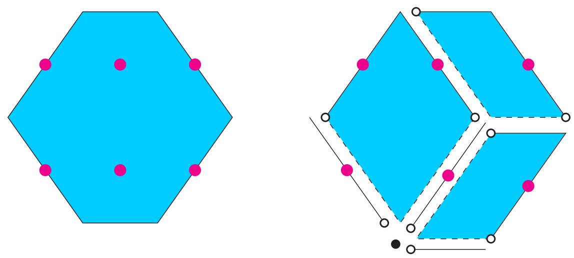

Figure 2: Decomposition of the fixed polytope $\Pi_4^{(12)}$ into half-open parallelotopes.

Example 4.1. The fixed polytope $\Pi_4^{(12)}$ of Figure 1, which corresponds to the cycle type $\lambda = (2, 1, 1)$, is

$$\Pi_4^{(12)} = [2\mathbf{e}_3, \mathbf{e}_{12}] + [2\mathbf{e}_4, \mathbf{e}_{12}] + [\mathbf{e}_4, \mathbf{e}_3] + \frac{3}{2}\mathbf{e}_{12} + \mathbf{e}_3 + \mathbf{e}_4.$$

Figure 2 shows its decomposition into parallelograms indexed by the forests on vertex set $\{12, 3, 4\}$. The three trees give parallelograms with volumes $2, 1, 1$ that contain $2, 1, 1$ lattice points, respectively. The three forests with one edge give edges of volumes $1, 1, 1$ and $1, 1, 0$ lattice points, respectively. The empty forest gives a point of volume 1 and 0 lattice points. Hence the Ehrhart quasipolynomial of $\Pi_4^{(12)}$ is

$$L_{\Pi_4^{(12)}}(t) = \begin{cases} (2+1+1)t^2 + (1+1+1)t + 1 & \text{if } t \text{ is even}\\ (2+1+1)t^2 + (1+1+0)t + 0 & \text{if } t \text{ is odd} \end{cases}.$$

Following the reasoning of Example 4.1, we will find the Ehrhart quasipolynomial of Π_n^σ by examining its decomposition into half-open parallelotopes. In order to find the number of lattice points in each parallelotope $\square_F$, the following observation is crucial.

Lemma 4.2. *[1, 7] If $\square$ is a lattice parallelotope in $\mathbb{Z}^n$ and $v \in \mathbb{Z}^n$, the number of lattice points in $\square + v$ is*

$$|(\square + v) \cap \mathbb{Z}^n| = \begin{cases} \mathrm{vol}(\square) & \textit{if the affine span of } \square + v \textit{ intersects the lattice } \mathbb{Z}^n\\ 0 & \textit{otherwise} \end{cases}.$$

We now apply Lemma 4.2 to the parallelotopes $\square_F$. Surprisingly, whether $\mathrm{aff}(\square_F)$ contains lattice points does not depend on the forest F, but only on the set partition π of the vertex set $[m]$ induced by the connected components of F. To make this precise we need a definition. Recall that the 2-*valuation* of a positive integer is the largest power of 2 dividing that integer; for example, $\mathrm{val}_2(24) = 3$.

Definition 4.3. *Let $\lambda = (\ell_1, \ldots, \ell_m)$ be a partition of the integer n. A set partition $\pi = \{B_1, \ldots, B_k\}$ of $[m]$ is called* λ-compatible *if for each block $B_i \in \pi$, at least one of the following conditions holds:*

(i) ℓ_j is odd for some $j \in B_i$, or

(ii) the minimum 2-valuation among $\{\ell_j : j \in B_i\}$ occurs an even number of times.

Example 4.4. Let $\lambda = (\ell_1, \ell_2, \ell_3)$ and $\mathrm{val}_2(\ell_i) = v_i$ for $i = 1, 2, 3$, and assume that $v_1 \geq v_2 \geq v_3$. Table 1 shows which partitions of [3] are λ-compatible depending on $\mathrm{val}_2(\lambda)$.

	123	12|3	13|2	23|1	1|2|3
$v_1 = v_2 = v_3 = 0$	•	•	•	•	•
$v_1 = v_2 = v_3 > 0$					
$v_1 = v_2 > v_3 = 0$	•	•			
$v_1 = v_2 > v_3 > 0$					
$v_1 > v_2 = v_3 = 0$	•	•	•		
$v_1 > v_2 = v_3 > 0$	•				
$v_1 > v_2 > v_3 = 0$	•				
$v_1 > v_2 > v_3 > 0$					

Table 1: λ-compatibility for $m = 3$.

Lemma 4.5. *Let $\sigma \in S_n$ have cycle type $\lambda = (\ell_1, \ldots, \ell_m)$. Let F be a forest on $[m]$ whose connected components induce the partition $\pi = \{B_1, \ldots, B_k\}$ of $[m]$. Then $\mathrm{aff}(\square_F)$ intersects the lattice $\mathbb{Z}^n$ if and only if π is λ-compatible.*

Proof. First we claim that

$$\mathrm{aff}(\square_F) = \left\{ \sum_{j=1}^{m} x_j \mathbf{e}_{\sigma_j} \; : \; \sum_{j \in B_i} \ell_j x_j = \sum_{j \in B_i} \frac{\ell_j(\ell_j+1)}{2} \text{ for } 1 \leq i \leq k \right\}. \tag{11}$$

Let $E(F)$ be the edge set of F. We have $\mathrm{aff}(\square_F) = \mathrm{span}\{\ell_b \mathbf{e}_{\sigma_a} - \ell_a \mathbf{e}_{\sigma_b} : \{a, b\} \in E(F)\} + \sum_{a=1}^{m} \frac{1}{2}(\ell_a + 1)\mathbf{e}_{\sigma_a}$. A point $\mathbf{y} \in \mathrm{span}\{\ell_b \mathbf{e}_{\sigma_a} - \ell_a \mathbf{e}_{\sigma_b} : \{a, b\} \in E(F)\}$ will satisfy $\sum_{j \in B_i} \ell_j y_j = 0$ for each block B_i. Furthermore, the translating vector $\mathbf{v} := \sum_{a=1}^{m} \frac{1}{2}(\ell_a + 1)\mathbf{e}_{\sigma_a}$ satisfies $\sum_{j \in B_i} \ell_j v_j = \sum_{j \in B_i} \frac{1}{2}\ell_j(\ell_j + 1)$ for each block B_i. Thus every point $\mathbf{x}$ in the affine span of $\square_F$ satisfies the given equations. These are all the relations among the x_js because each block B_i contributes $|E(B_i)| = |B_i| - 1$ to the dimension of the affine span of $\square_F$.

This affine subspace intersects the lattice $\mathbb{Z}^n$ if and only if all equations in (11) have integer solutions. Elementary number theory tells us that this is the case if and only if each block B_i satisfies

$$\gcd(\ell_j : j \in B_i) \;\Bigg|\; \sum_{j \in B_i} \frac{\ell_j(\ell_j+1)}{2}. \tag{12}$$

It is always true that $\gcd(\ell_j : j \in B_i)$ divides $\sum_{j \in B_i} \ell_j(\ell_j + 1)$, so (12) holds if and only if

$$\mathrm{val}_2\Big(\gcd(\ell_j : j \in B_i)\Big) < \mathrm{val}_2\Big(\sum_{j \in B_i} \ell_j(\ell_j + 1)\Big). \tag{13}$$

We consider two cases.

(i) Suppose ℓ_j is odd for some $j \in B_i$. Then $\gcd(\ell_j : j \in B_i)$ is odd, whereas $\sum_{j\in B_i} \ell_j(\ell_j+1)$ is always even. Hence (13) always holds in this case.

(ii) Suppose that ℓ_j is even for all $j \in B_i$. For each ℓ_j, write $\ell_j = 2^{p_j} q_j$ for some integer $p_j \geq 1$ and odd integer q_j. Then $\mathrm{val}_2(\gcd(\ell_j : j \in B_i)) = \min_{j\in B_i} p_j$; we will call this integer p. We have

$$\begin{aligned}\mathrm{val}_2\Big(\sum_{j\in B_i} \ell_j(\ell_j+1)\Big) &= \mathrm{val}_2\Big(\sum_{j\in B_i} 2^{p_j} q_j(\ell_j+1)\Big)\\ &= p + \mathrm{val}_2\Big(\sum_{j\in B_i} 2^{p_j-p} q_j(\ell_j+1)\Big).\end{aligned}$$

Note that $q_j(\ell_j+1)$ is odd for each j. If the minimum 2-valuation p of $\{\ell_j : j \in B_i\}$ occurs an odd number of times, then $\sum_{j\in B_i} 2^{p_j-p} q_j(\ell_j+1)$ will be odd and we will have $\mathrm{val}_2(\sum_{j\in B_i} \ell_j(\ell_j+1)) = p$. Otherwise, this sum will be even and we will have $\mathrm{val}_2(\sum_{j\in B_i} \ell_j(\ell_j+1)) > p$. Therefore (13) holds if and only if the minimum 2-valuation among the ℓ_j for $j \in B_i$ occurs an even number of times. This is precisely the condition of λ-compatibility. $\square$

We now have all of the necessary tools to compute the Ehrhart quasipolynomial of the fixed polytope Π_n^σ. Recall the definition of λ-compatibility in 4.3 and the definition of v_π in (1).

Theorem 1.1. *Let σ be a permutation of $[n]$ with cycle type $\lambda = (\ell_1, \ldots, \ell_m)$. Then the Ehrhart quasipolynomial of the fixed polytope of the permutahedron Π_n fixed by σ is*

$$L_{\Pi_n^\sigma}(t) = \begin{cases} \displaystyle\sum_{\pi \vDash [m]} v_\pi \cdot t^{m-|\pi|} & \text{if } t \text{ is even}\\ \displaystyle\sum_{\substack{\pi \vDash [m]\\ \lambda-\text{compatible}}} v_\pi \cdot t^{m-|\pi|} & \text{if } t \text{ is odd}\end{cases}.$$

Proof. We calculate the number of lattice points in each integer dilate $t\Pi_n^\sigma$ by decomposing it into half-open parallelotopes and adding up the number of lattice points inside of each parallelotope.

First, suppose that t is even. Then $t\Pi_n^\sigma$ is a lattice polytope, all parallelotopes in the decomposition of $t\Pi_n^\sigma$ have vertices on the integer lattice, and each i-dimensional parallelotope $\square$ contains $\mathrm{vol}(\square)t^i$ lattice points [3, Lemma 9.2]. The parallelotopes correspond to linearly independent subsets of the vector configuration $\{\ell_i \mathbf{e}_{\sigma_j} - \ell_j \mathbf{e}_{\sigma_i} : 1 \leq i < j \leq m\}$, which is in bijection with forests on $[m]$. Following the reasoning used to prove Theorem 3.1, we conclude that when t is even,

$$L_{\Pi_n^\sigma}(t) = \sum_{\pi \vDash [m]} v_\pi \cdot t^{m-|\pi|}.$$

Next, suppose t is odd. Then $t\Pi_n^\sigma$ is half-integral, but it may not be a lattice polytope. As before, we may decompose $t\Pi_n^\sigma$ into half-open parallelotopes that are in bijection with forests on $[m]$. Lemma 4.2, Lemma 4.5, and [3, Lemma 9.2] tell us that $\square_F$ contains $\mathrm{vol}(\square_F)t^{m-|\pi|}$ lattice points if the set partition π induced by F is λ-compatible, and 0 otherwise. Therefore if t is odd

$$L_{\Pi_n^\sigma}(t) = \sum_{\substack{\pi \vDash [m]\\ \lambda-\text{compatible}}} v_\pi \cdot t^{m-|\pi|}$$

as desired. $\square$

5 The equivariant H^*-series of the permutahedron

We now compute the equivariant H^*-series of the permutahedron and characterize when it is polynomial and when it is effective, proving Stapledon's Effectiveness Conjecture 1.2 in this special case.

The *Ehrhart series* of a rational polytope P is

$$\mathrm{Ehr}_P(z) = \sum_{t=0}^{\infty} L_P(t) \cdot z^t.$$

In computing the Ehrhart series of Π_n^σ, Eulerian polynomials naturally arise. The *Eulerian polynomial* $A_k(z)$ is defined by the identity

$$\sum_{t \geq 0} t^k z^t = \frac{A_k(z)}{(1-z)^{k+1}}.$$

Proposition 5.1. *Let $\sigma \in S_n$ have cycle type $\lambda = (\ell_1, \ldots, \ell_m)$. The Ehrhart series of Π_n^σ is*

$$\mathrm{Ehr}_{\Pi_n^\sigma}(z) = \sum_{\substack{\pi \vDash [m] \\ \lambda\text{-compatible}}} \frac{v_\pi \cdot A_{m-|\pi|}(z)}{(1-z)^{m-|\pi|+1}} + \sum_{\substack{\pi \vDash [m] \\ \lambda\text{-incompatible}}} \frac{v_\pi \cdot 2^{m-|\pi|} \cdot A_{m-|\pi|}(z^2)}{(1-z^2)^{m-|\pi|+1}}$$

and the H^*-series of the permutahedron equals

$$H^*[z](\sigma) = \left(\prod_{i=1}^{m} (1 - z^{\ell_i}) \right) \cdot \mathrm{Ehr}_{\Pi_n^\sigma}(z)$$

Proof. The first statement follows readily from Theorem 1.1:

$$\begin{aligned}
\mathrm{Ehr}_{\Pi_n^\sigma}(z) &= \sum_{t \text{ even}} \left(\sum_{\pi \vDash [m]} v_\pi t^{m-|\pi|} \right) z^t + \sum_{t \text{ odd}} \left(\sum_{\substack{\pi \vDash [m] \\ \lambda\text{-compatible}}} v_\pi t^{m-|\pi|} \right) z^t \\
&= \sum_{\substack{\pi \vDash [m] \\ \lambda\text{-compatible}}} v_\pi \left(\sum_{t=0}^{\infty} t^{m-|\pi|} z^t \right) + \sum_{\substack{\pi \vDash [m] \\ \lambda\text{-incompatible}}} v_\pi \left(\sum_{t \text{ even}} t^{m-|\pi|} z^t \right) \\
&= \sum_{\substack{\pi \vDash [m] \\ \lambda\text{-compatible}}} v_\pi \frac{A_{m-|\pi|}(z)}{(1-z)^{m-|\pi|+1}} + \sum_{\substack{\pi \vDash [m] \\ \lambda\text{-incompatible}}} v_\pi \cdot 2^{m-|\pi|} \frac{A_{m-|\pi|}(z^2)}{(1-z^2)^{m-|\pi|+1}}
\end{aligned}$$

For the second statement, recall that $H^*[z]$ is defined as in (2), where ρ is the standard representation of S_n in this case. The left hand side is the Ehrhart series. The denominator on the right side is $(1-z)\det(I - \rho(\sigma) \cdot z)$; it equals the characteristic polynomial of the permutation matrix of σ, which is $\prod_{i=1}^{m}(1 - z^{\ell_i})$. ☐

Tables 2 and 3 show the equivariant H^*-series of the permutahedra Π_3 and Π_4.

Stapledon writes that *"The main open problem is to characterize when $H^*[z]$ is effective"*, and he conjectures the following characterization:

Conjecture 1.2 ([11, Effectiveness Conjecture 12.1])**.** *Let P be a lattice polytope invariant under the action of a group G. The following conditions are equivalent.*

(i) The toric variety of P admits a G-invariant non-degenerate hypersurface.

(ii) The equivariant H^*-series of P is effective.

(iii) The equivariant H^*-series of P is a polynomial.

He shows that (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii), so only the reverse implications are conjectured. Our next goal is to verify Stapledon's conjecture for the action of S_n on the permutahedron Π_n. We do so by showing that the conditions of Conjecture 1.2 hold if and only if $n \leq 3$.

Cycle type of $\sigma \in S_3$	$\chi_{t\Pi_3}(\sigma)$	$\sum_{t\geq 0} \chi_{t\Pi_3}(\sigma) z^t$	$H^*[z](\sigma)$
$(1,1,1)$	$3t^2+3t+1$	$\dfrac{1+4z+z^2}{(1-z)^3}$	$1+4z+z^2$
$(2,1)$	$\begin{cases} t+1 & \text{if } t \text{ is even} \\ t & \text{if } t \text{ is odd} \end{cases}$	$\dfrac{1+z^2}{(1-z)(1-z^2)}$	$1+z^2$
(3)	1	$\dfrac{1}{1-z} = \dfrac{1+z+z^2}{1-z^3}$	$1+z+z^2$

Table 2: The equivariant H^*-series of Π_3

Cycle type of $\sigma \in S_4$	$\chi_{t\Pi_4}(\sigma)$	$\sum_{t\geq 0} \chi_{t\Pi_4}(\sigma) z^t$	$H^*[z](\sigma)$
$(1,1,1,1)$	$16t^3+15t^2+6t+1$	$\dfrac{1+34z+55z^2+6z^3}{(1-z)^4}$	$1+34z+55z^2+6z^3$
$(2,1,1)$	$\begin{cases} 4t^2+3t+1 & \text{if } t \text{ is even} \\ 4t^2+2t & \text{if } t \text{ is odd} \end{cases}$	$\dfrac{1+6z+20z^2+24z^3+11z^4+2z^5}{(1-z)^2(1-z^2)(1+z)^2}$	$1+4z+11z^2-2z^3+\sum_{i=4}^{\infty} 4(-1)^i z^i$
$(3,1)$	$t+1$	$\dfrac{1}{(1-z)^2} = \dfrac{1+z+z^2}{(1-z)(1-z^3)}$	$1+z+z^2$
(4)	$\begin{cases} 1 & \text{if } t \text{ is even} \\ 0 & \text{if } t \text{ is odd} \end{cases}$	$\dfrac{1}{1-z^2} = \dfrac{1+z^2}{1-z^4}$	$1+z^2$
$(2,2)$	$\begin{cases} 2t+1 & \text{if } t \text{ is even} \\ 2t & \text{if } t \text{ is odd} \end{cases}$	$\dfrac{1+2z+3z^2+2z^3}{(1-z^2)^2}$	$1+2z+3z^2+2z^3$

Table 3: The equivariant H^*-series of Π_4

5.1 Polynomiality of $H^*[z]$

Lemma 5.2. *Let $\sigma \in S_n$ have cycle type $\lambda = (\ell_1, \ldots, \ell_m)$. The equivariant H^*-series evaluated at σ, $H^*[z](\sigma)$, is a polynomial if and only if the number of even parts in λ is 0, $m-1$, or m.*

Proof. By Proposition 5.1, the Ehrhart series $\mathrm{Ehr}_{\Pi_n^\sigma}(z)$ may only have poles at $z = \pm 1$. The pole at $z = 1$ has order at most m. Since the polynomial $\prod_{i=1}^m (1 - z^{\ell_i})$ has a zero at $z = 1$ of order m, the series $H^*[z](\sigma)$ will not have a pole at $z = 1$. Hence we only need to check whether $H^*[z](\sigma)$ has a pole at $z = -1$.

(i) First, suppose no ℓ_i is even. Then all partitions of $[m]$ are λ-compatible, so $\mathrm{Ehr}_{\Pi_n^\sigma}(z)$ does not have a pole at $z = -1$. Thus $H^*[z](\sigma)$ is a polynomial in this case.

(ii) Next, suppose that some ℓ_i is even. Then the partition $\{\{\ell_i\}, [m] - \{\ell_i\}\}$ is λ-incompatible, so $\mathrm{Ehr}_{\Pi_n^\sigma}(z)$ does have a pole at $z = -1$. It is well known that $A_k(1) = k!$ so every numerator $v_\pi \cdot 2^{m-|\pi|} \cdot A_{m-|\pi|}(z^2)$ is positive at $z = -1$. It follows that the order of the pole $z = -1$ of $\mathrm{Ehr}_{\Pi_n^\sigma}(z)$ is $m - d + 1$ where $d = \min\{|\pi| : \pi \text{ is } \lambda\text{-incompatible}\}$. This equals $m - 1$ if the partition $\{[m]\}$ is λ-compatible and m if it is λ-incompatible.

On the other hand, $\prod_{i=1}^m (1 - z^{\ell_i})$ has a zero at $z = -1$ of order equal to the number of even ℓ_i. Now consider three cases:

a) If the number of even ℓ_i is between 1 and $m-2$, it is less than the the order of the pole of $\mathrm{Ehr}_{\Pi_n^\sigma}(z)$, so $H^*[z](\sigma)$ is not polynomial.

b) If all ℓ_i are even, the zero $z = -1$ in $\prod_{i=1}^m (1 - z^{\ell_i})$ has order m and cancels the pole in $\mathrm{Ehr}_{\Pi_n^\sigma}(z)$. Thus $H^*[z](\sigma)$ is polynomial.

c) If $m-1$ of the ℓ_i are even, the partition $\{[m]\}$ is λ-compatible. Therefore the order of the pole in $\mathrm{Ehr}_{\Pi_n^\sigma}(z)$ and the order of the zero in $\prod_{i=1}^{m}(1-z^{\ell_i})$ both equal $m-1$, and $H^*[z](\sigma)$ is polynomial. $\square$

Proposition 5.3. *The equivariant H^*-series of the permutahedron Π_n is a polynomial if and only if $n \leq 3$.*

Proof. When $n \leq 3$, all partitions of n have 0, 1, or all odd parts. Hence $H^*[z](\sigma)$ is a polynomial for all $\sigma \in S_n$, so $H^*[z]$ is a polynomial.

Suppose $n \geq 4$. Then there always exists some partition of n with more than 1 but fewer than all odd parts: if n is even we can take the partition $(n-2,1,1)$, and if n is odd we can take the partition $(n-3,1,1,1)$. Therefore $H^*[z]$ is not polynomial. $\square$

5.2 Effectiveness of $H^*[z]$

Proposition 5.4. *The equivariant H^*-series of the permutahedron Π_n is effective if and only if $n \leq 3$.*

Proof. Stapledon [11] observed that if H^* is effective then it is polynomial. Thus by Proposition 5.3 we only need to check effectiveness for $n = 1, 2, 3$.

Let us check it for $n = 3$. Table 2 shows that $H^*[z] = H_0^* + H_1^* z + H_2^* z^2$ for $H_0^*, H_1^*, H_2^* \in R(S_3)$. Comparing these with the character table of S_3 (see for example [5, pg.14]) gives

$$H_0^* = \chi_{triv}, \qquad H_1^* = \chi_{triv} + \chi_{alt} + \chi_{std}, \qquad H_2^* = \chi_{triv}.$$

Since all coefficients are nonnegative, $H^*_{\Pi_3}[z] = \chi_{triv} + (\chi_{triv} + \chi_{alt} + \chi_{std})z + \chi_{triv} z^2$ is indeed effective.

Similarly, $H^*_{\Pi_2}[z] = \chi_{triv}$ and $H^*_{\Pi_1}[z] = \chi_{triv}$ are effective as well. $\square$

In contrast, a similar computation based on Table 3 gives

$$\begin{aligned} H^*_{\Pi_4} &= \chi_{triv} + (3\chi_{triv} + \chi_{alt} + 5\chi_{std} + 3\chi_{\ydiagram{2,1,1}} + 3\chi_{\ydiagram{2,2}})z + (6\chi_{triv} + 9\chi_{std} + 4\chi_{\ydiagram{2,1,1}} + 5\chi_{\ydiagram{2,2}})z^2 \\ &+ (\chi_{alt} + \chi_{\ydiagram{2,1,1}} + \chi_{\ydiagram{2,2}})z^3 + (\chi_{triv} - \chi_{alt} + \chi_{std} - \chi_{\ydiagram{2,1,1}})(z^4 - z^5 + z^6 - z^7 + \cdots) \end{aligned}$$

which is not effective.

5.3 S_n-invariant non-degenerate hypersurfaces in the permutahedral variety

We begin by explaining condition (i) of Conjecture 1.2, which arises from Khovanskii's notion of non-degeneracy [6]. We refer the reader to [11, Section 7] for more details.

Let $P \subset \mathbb{R}^n$ be a lattice polytope with an action of a finite group G. For $v \in \mathbb{Z}^n$ we write $x^v := x_1^{v_1} \cdot \ldots \cdot x_n^{v_n}$. The coordinate ring of the projective toric variety X_P of P has the form $\mathbb{C}[x^v : v \in P \cap \mathbb{Z}^n]$, so a hypersurface in X_P is given by a linear equation $\sum_{v \in P \cap \mathbb{Z}^n} a_v x^v = 0$ for some complex coefficients a_v. The group G acts on the monomials x^v by its action on the lattice points $v \in P \cap \mathbb{Z}^n$, so the equation of a G-invariant hypersurface should have $a_v = a_u$ whenever u and v are in the same G-orbit. A projective hypersurface in X_P with equation $f(x_1, \ldots, x_n) = 0$ is *smooth* if the gradient $(\partial f/\partial x_1, \ldots, \partial f/\partial x_n)$ is never zero when $(x_1, \ldots, x_n) \in (\mathbb{C}^*)^n$. There is a unique polynomial in the a_vs, called the *discriminant*, such that the hypersurface is smooth when the discriminant does not vanish at the coefficients a_v. A hypersurface in the toric variety of P is *non-degenerate* if it is smooth and for each face F of P, the hypersurface $\sum_{v \in F \cap \mathbb{Z}^n} a_v x^v = 0$ is also smooth.

The *permutahedral variety* X_{Π_n} is the projective toric variety associated to the permutahedron Π_n.

Proposition 5.5. *The permutahedral variety X_{Π_n} admits an S_n-invariant non-degenerate hypersurface if and only if $n \leq 3$.*

Proof. Stapledon proved [12, Theorem 7.7] that if X_{Π_n} admits such a hypersurface, then $H^*[z]$ is effective. By Proposition 5.4, this can only occur for $n = 1, 2, 3$.

Case 1: $n = 1$.

A hypersurface in the toric variety of $\Pi_1 = \{1\} \subset \mathbb{R}$ has the form $ax = 0$, and since we are working over projective space, we can assume $a = 1$. The derivative of this never vanishes, so this is a smooth S_1-invariant hypersurface.

Case 2: $n = 2$.

The permutahedron Π_2 is the line segment with vertices $(1, 2), (2, 1) \in \mathbb{R}^2$ and no other lattice points. The vertices are in the same S_2-orbit, so we need to check that hypersurface with equation $xy^2 + x^2y = 0$ is non-degenerate. The gradient is $(y(y + 2x), x(2y + x))$, which never vanishes on $(\mathbb{C}^*)^2$. The vertex $(1, 2)$ corresponds to the hypersurface $xy^2 = 0$. The gradient of this is $(y^2, 2xy)$ which also never vanishes on $(\mathbb{C}^*)^2$. The computation for the other vertex is similar. Hence this is an S_2-invariant non-degenerate hypersurface.

Case 3: $n = 3$.

The permutahedron Π_3 is a hexagon with one interior point. Choosing the vertices to be all permutations of the point $(0, 1, 2) \in \mathbb{R}^3$ (instead of $(1, 2, 3)$) will simplify calculations. The six vertices of the hexagon are one S_3-orbit and the interior point is its own orbit. Hence (up to scaling) an S_3-invariant hypersurface must have the equation

$$a \cdot xyz + yz^2 + y^2z + xy^2 + x^2y + xz^2 + x^2z = 0 \tag{14}$$

which has one parameter a. We want to check whether there exists some choice of a for which this hypersurface is non-degenerate. We need to check this on each face.

The vertex $(0, 1, 2)$ gives the hypersurface $yz^2 = 0$ with gradient $(0, z^2, 2yz)$. This never vanishes on $(\mathbb{C}^*)^3$, so it is smooth. The computations for the other five vertices are similar.

For the edge connecting $(0, 1, 2)$ and $(0, 2, 1)$, the corresponding hypersurface is $yz^2 + y^2z = 0$. This is the same hypersurface as the line segment Π_2, so it is smooth; so are the hypersurfaces of the other five edges.

Finally, we need to show there exists a such that the entire hypersurface is smooth. This is the same as showing that the discriminant of (14) is not identically zero. Since (14) is a symmetric polynomial, we can write in terms of the power-sum symmetric polynomials, $p_k = x^k + y^k + z^k$; we obtain

$$\frac{a}{6}p_1^3 + \left(1 - \frac{a}{2}\right) p_1p_2 + \left(\frac{a}{3} - 1\right) p_3 = 0. \tag{15}$$

The discriminant of a degree 3 symmetric polynomial is given in [8, Equation 64]; substituting the coefficients $a/6$, $1 - a/2$, and $a/3 - 1$ gives a non-zero polynomial of degree 12:

$$\begin{gathered}\frac{-512000}{16677181699666569}a^{12} + \frac{492800}{617673396283947}a^{10} - \frac{985600}{617673396283947}a^9 + \frac{6320}{7625597484987}a^8 \\ - \frac{25280}{7625597484987}a^7 + \frac{27431}{7625597484987}a^6 - \frac{478}{282429536481}a^5 + \frac{965}{282429536481}a^4 \\ - \frac{2128}{847288609443}a^3 + \frac{8}{10460353203}a^2 - \frac{32}{31381059609}a + \frac{16}{31381059609}\end{gathered}$$

Any value of a that is not a root of this discriminant gives us an S_3-invariant non-degenerate hypersurface. □

By contrast, we should not be able to find an S_n-invariant non-degenerate hypersurface in X_{Π_n} for $n \geq 4$. This can be seen from the fact that all permutahedra Π_n when $n \geq 4$ have a square face, and the hypersurface of this square face is not smooth. For example, consider the square face of Π_4 with vertices $(0, 1, 2, 3)$, $(0, 1, 3, 2)$, $(1, 0, 3, 2)$, and $(1, 0, 2, 3)$. The corresponding hypersurface is $yz^2w^3 + yz^3w^2 + xz^3w^2 + xz^2w^3 = 0$, and its gradient vanishes whenever $x = -y$ and $z = -w$.

5.4 Stapledon's Conjectures

Our second main result now follows as a corollary.

Theorem 1.3. *Stapledon's Effectiveness Conjecture holds for the permutahedron under the action of the symmetric group.*

Proof. This follows immediately from Propositions 5.3, 5.4, and 5.5 □

In closing, we verify the remaining three conjectures of Stapledon for the special case of the S_n–action on the permutahedron Π_n.

Conjecture 5.6. *[11, Conjecture 12.2] If $H^*[z]$ is effective, then $H^*[1]$ is a permutation representation.*

Conjecture 5.7. *[11, Conjecture 12.3] For a polytope $P \subset \mathbb{R}^n$, let $\operatorname{ind}(P)$ be the smallest positive integer k such that the affine span of kP contains a lattice point. For any $g \in G$, let M^g be the sublattice of M fixed by g, and define $\det(I-\rho(g))_{(M^g)^\perp}$ to be the determinant of $I-\rho(g)$ when the action of $\rho(g)$ is restricted to $(M^g)^\perp$. The quantity*

$$H^*[1](g) = \frac{\dim(P^g)! \cdot \operatorname{vol}(P^g) \cdot \det(I-\rho(g))_{(M^g)^\perp}}{\operatorname{ind}(P^g)}$$

is a non-negative integer.

Conjecture 5.8. *[11, Conjecture 12.4] If $H^*[z]$ is a polynomial and the i^{th} coefficient of the h^*-polynomial of P is positive, then the trivial representation occurs with non-zero multiplicity in the virtual character H_i^*.*

Proposition 5.9. *Conjectures 5.6, 5.7, and 5.8 hold for permutahedra under the action of the symmetric group.*

Proof. 5.6: This statement only applies to Π_1, Π_2, and Π_3. From the proof of Proposition 5.4 we obtain that $H^*[1]$ is the trivial character for Π_1 and Π_2 and the statement holds. For Π_3 we have

$$H^*[1] = 3\chi_{triv} + \chi_{alt} + \chi_{std} = \chi_{triv} + (\chi_{triv} + \chi_{alt}) + (\chi_{triv} + \chi_{std}). \tag{16}$$

Now $\chi_{triv} + \chi_{alt}$ is the permutation character of the sign action of S_3 on the set $[2]$, and $\chi_{triv} + \chi_{std}$ is the character of the permutation representation of S_3. Hence all summands on the right side of (16) are permutation characters, so their sum is as well.

5.7: For $\sigma \in S_n$ of cycle type $\lambda = (\ell_1, \ldots, \ell_m)$ we have $\dim(\Pi_n^\sigma) = m-1$ and $\operatorname{vol}(\Pi_n^\sigma) = n^{m-2}\gcd(\ell_1, \ldots, \ell_m)$. Now, the fixed lattice $M^g = \mathbb{Z}\{\mathbf{e}_{\sigma_1}, \cdots, \mathbf{e}_{\sigma_m}\}$ has rank m, so

$$\det(I-\rho(\sigma)\cdot z)_{(M^\sigma)^\perp} = \frac{(1-z)\det(I-\rho(\sigma)\cdot z)}{(1-z)^m} = \prod_{i=1}^m (1+z+\cdots+z^{\ell_i-1}).$$

Therefore the numerator is $(m-1)! \cdot n^{m-2} \cdot \gcd(\ell_1, \ldots, \ell_m) \cdot \ell_1 \cdots \ell_m$. The denominator is

$$\operatorname{ind}(\Pi_n^\sigma) = \begin{cases} 2 & \text{if all } \pi \vDash [m] \text{ are } \lambda\text{-incompatible,} \\ 1 & \text{otherwise.} \end{cases}$$

When the denominator is 2, all the ℓ_i must be even, so the numerator is even. The desired result follows.

5.8: We need to check this for Π_1, Π_2, and Π_3. For Π_1 and Π_2 the h^*-polynomial is 1 and $H_0^* = \chi_{triv}$. For Π_3, the h^*-polynomial is $1+4z+z^2$, and $H_0^* = \chi_{triv}$, $H_1^* = \chi_{triv} + \chi_{alt} + \chi_{std}$, and $H_2^* = \chi_{triv}$ all contain a copy of the trivial character. □

6 Acknowledgments

We would like to thank Sophia Elia for developing a useful Sage package for equivariant Ehrhart theory, and Matthias Beck, Benjamin Braun, Christopher Borger, Ana Botero, Sophia Elia, Donghyun Kim, Jodi McWhirter, Dusty Ross, Kristin Shaw, and Anna Schindler for fruitful conversations. This work was completed while FA was a Spring 2019 Visiting Professor at the Simons Institute for Theoretical Computer Science in Berkeley, and a 2019-2020 Simons Fellow while on sabbatical in Bogotá; he is very grateful to the Simons Foundation, San Francisco State University, and the Universidad de Los Andes for their support.

References

[1] Federico Ardila, Matthias Beck, and Jodi McWhirter, *Ehrhart quasipolynomials of Coxeter permutahedra*.

[2] Federico Ardila, Anna Schindler, and Andrés R. Vindas-Meléndez, *The equivariant volumes of the permutahedron*, Discrete and Computational Geometry (2019), 1–18.

[3] Matthias Beck and Sinai Robins, *Computing the Continuous Discretely*, Undergraduate Texts in Mathematics, Springer, New York, 2007, Integer-point enumeration in polyhedra. MR 2271992 (2007h:11119)

[4] Eugène Ehrhart, *Sur les polyèdres rationnels homothétiques à n dimensions*, C. R. Acad. Sci. Paris **254** (1962), 616–618.

[5] William Fulton and Joe Harris, *Representation Theory*, Graduate Texts in Mathematics, vol. 129, Springer-Verlag, New York, 1991, A first course, Readings in Mathematics. MR 1153249 (93a:20069)

[6] Askold G Khovanskii, *Newton polyhedra and toroidal varieties*, Functional analysis and its applications **11** (1977), no. 4, 289–296.

[7] Jodi McWhirter, *Ehrhart quasipolynomials of Coxeter permutahedra*, Master's thesis, San Francisco State University, 2019.

[8] N. Perminov and Sh. Shakirov, *Discriminants of symmetric polynomials*, arXiv e-prints (2009), arXiv:0910.5757.

[9] Geoffrey C. Shephard, *Combinatorial properties of associated zonotopes*, Canad. J. Math. **26** (1974), 302–321.

[10] Richard P. Stanley, *A zonotope associated with graphical degree sequences*, Applied geometry and discrete mathematics, DIMACS Ser. Discrete Math. Theoret. Comput. Sci., vol. 4, Amer. Math. Soc., Providence, RI, 1991, pp. 555–570. MR 1116376 (92k:52020)

[11] Alan Stapledon, *Equivariant Ehrhart theory*, Advances in Mathematics **226** (2011), no. 4, 3622–3654.

[12] ———, *Representations on the cohomology of hypersurfaces and mirror symmetry*, Advances in Mathematics **226** (2011), no. 6, 5268–5297.