Tilings

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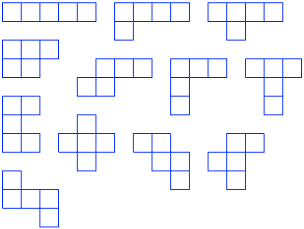
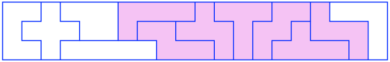
Definición: A tiling is a way of covering a given region using a given set of tiles completely and without any overlap.



For example, there are two ways of tiling a 3×20 board with the 12 pentominoes:

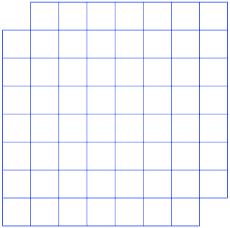


Outline:

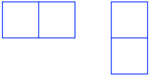
1. Graph theory.
2. Number theory.
3. Algebra.
4. Enumeration.
5. Probability.
6. Geometry.

1. Does a tiling exist?

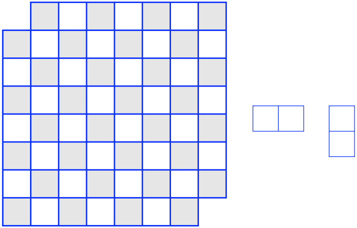
An easier, but more (mathematically) interesting question:
Remove two opposite corners from a chessboard.



Question. Can we tile this board with 31 dominoes?



A “chessboard" would not be a chessboard without its colors.

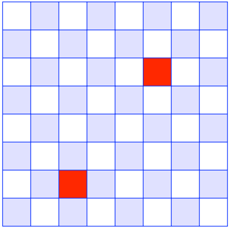


Answer. It is not possible, because

- any domino would **match** a white cell and a black cell, and
- the board has 30 white cells and 32 black cells.

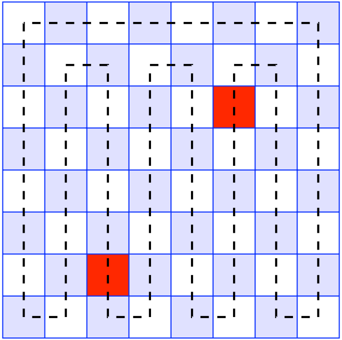
This is the simplest example of a *coloring argument*.

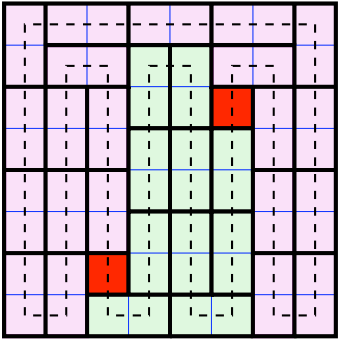
Question. If I remove one black and one white square from a chessboard, can you tile the resulting board with dominoes?



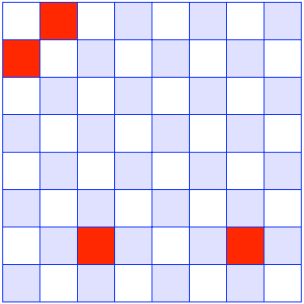
Question. If I remove one black and one white square from a chessboard, can you tile the resulting board with dominoes?

Answer. Yes, for **any** black and white squares.





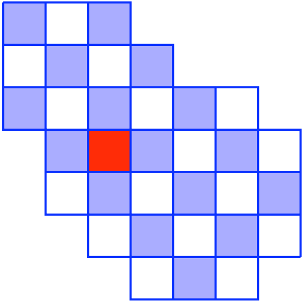
What if I eliminate two black and two white squares?



Sometimes a tiling exists, sometimes it doesn't.

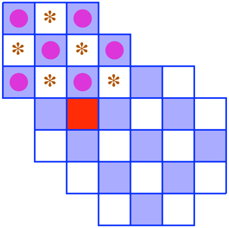
Question. If I remove k black squares and k white squares from a chessboard, how can you tell whether it is possible to tile the resulting board with dominoes?

For instance, this board has 16 black and 16 white cells. Can you tile it with dominoes?



You cannot tile this region with dominoes.
One explanation: A “deficient" set of cells, a sort of “bottleneck"

The 6 black cells marked ● are adjacent to only 5 white cells, labelled $*$!



Idea. “Tile" = “Marriage". Like my grandparents' village, it seems that people only want to marry one of their neighbors. (No poligamy in Colombia, and gay marriage is not legal yet.)

The Key: If we want everyone to get married, for any *k* men we need at least *k* women willing to marry one of them.

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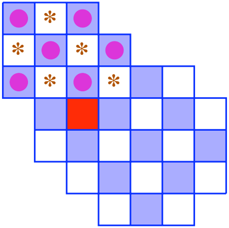

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In fact, this is the only possible explanation!

Marriage Theorem. (Hall, 1935.)
A board can be tiled with dominoes if and only if any k squares have at least k neighbors.

This is the beginning of matching theory – a very important field of combinatorial optimization, with many practical applications.

(Workers and jobs, students and universities)

Theorem. (Hall, 1935.)
It's **easy** to determine whether a region can be tiled with dominoes.
It's also easy to construct a tiling if one exists.

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But:

This question is very hard for almost any other tiling problem!

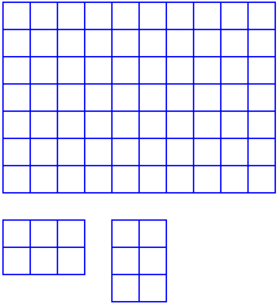
For the next simplest tile:

Theorem. (Beauquier et al, 1995.)
It's **hard** to determine whether a region can be tiled with 1×3 rectangles. There is no efficient algorithm to do this.

How hard? NP-hard. In fact, NP-complete.

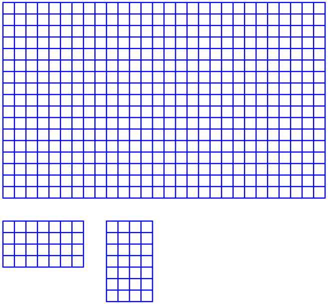
Number theoretic restrictions.

1. Can a 7×10 rectangle be tiled with 2×3 rectangles?



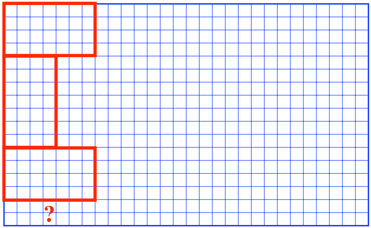
No. A 2×3 rectangle has 6 squares, while a 7×10 rectangle has 70 squares (not divisible by 6).

2. Can a 17×28 rectangle be tiled with 4×7 rectangles?



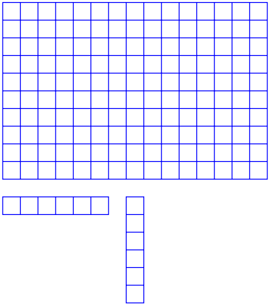
At least $28 \cdot 17$ is a multiple of $4 \cdot 7$.

2. Can a 17×28 rectangle be tiled with 4×7 rectangles?



No: there is no way to cover the first column, because one cannot write 17 as a sum of 4s and 7s.

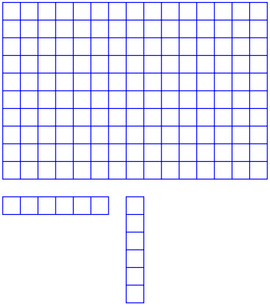
3. Can a 10×15 rectangle be tiled with 1×6 rectangles?



At least:

- $10 \cdot 15$ is a multiple of $1 \cdot 6$.
- $10 = 6 + 1 + 1 + 1 + 1$ and $15 = 6 + 6 + 1 + 1 + 1$.

3. Can a 10×15 rectangle be tiled with 1×6 rectangles?



No, because neither 10 nor 15 are divisible by 6.

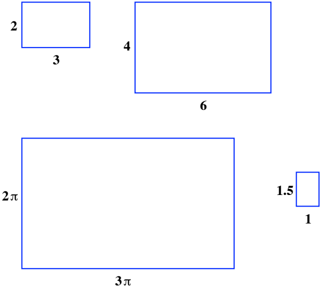
The full answer:

Theorem. (de Bruijn-Klarner, 1969)
An $m \times n$ rectangle can be tiled with $a \times b$ rectangles if and only if:

- The first row and first column can be covered.
- m or n is divisible by a, and m or n is divisible by b.

Algebraic restrictions.

Question.
For which $x > 0$ is it possible to tile a square with rectangles similar to the $1 \times x$ rectangle?



For example, is it possible for $1 \times \sqrt{2}$?

Algebraic restrictions.

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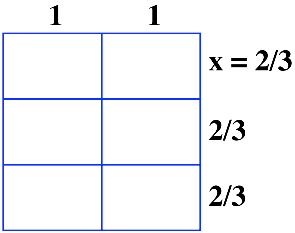
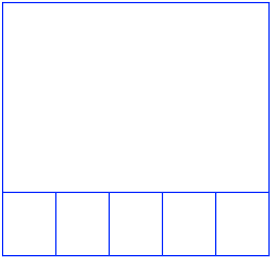
A boring case:



For $x = \frac{2}{3}$ (or any positive rational) it is possible.

We could also lay out a tiling, and find the value of x:



$$\frac{1}{x} = \frac{1-x}{1/5}$$

$$x(1-x) = \frac{1}{5}$$

$$5x^2 - 5x + 1 = 0$$

$$x = \frac{5+\sqrt{5}}{10} = 0.7236\ldots$$

Note that the other root also works:

$$x = \frac{5-\sqrt{5}}{10} = 0.2763\ldots$$

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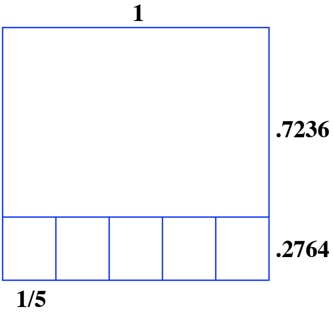

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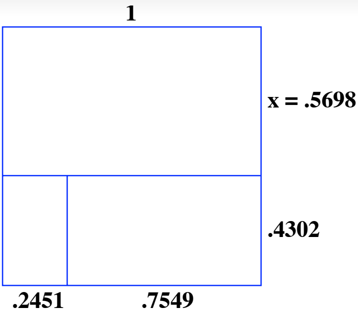


$$x^3 - x^2 + 2x - 1 = 0$$

$$x = 0.5698\ldots$$

The other two roots don't **really** give solutions:

$$x = 0.215\ldots + 1.307\ldots\sqrt{-1}$$

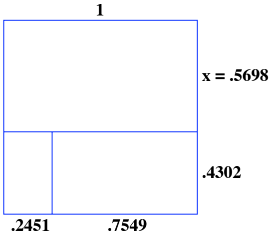
$$x = 0.215\ldots - 1.307\ldots\sqrt{-1}$$

What about $x = \pi$? Or $x = \sqrt{2}$?

The full answer:

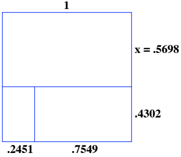
Theorem. (Freiling-Rinne, Laczkovich-Szekeres, 1995)
Let $x > 0$. It is possible to tile a square with rectangles similar to the $1 \times x$ rectangle if and only if:

1. x is **algebraic**: There exists a (minimal) polynomial $P(z)$ with integer coefficients such that $P(x) = 0$, and
2. every (complex) solution of the equation $P(z) = 0$ has positive real part.



Examples.

- $x = 0.5698\ldots$
  Min. polynomial: $x^3 - x^2 + 2x - 1 = 0$
  Other roots: $x = \mathbf{0.215\ldots} \pm 1.307\ldots\sqrt{-1}$



- $x = \sqrt{2}$.
  Minimal polynomial: $x^2 - 2 = 0$
  Other root: $-\sqrt{2} < 0$.

Theorem. (Freiling-Rinne, Laczkovich-Szekeres, 1995)
It is **not possible** to tile a square with rectangles similar to the $1 \times \sqrt{2}$ rectangle.

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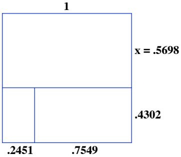

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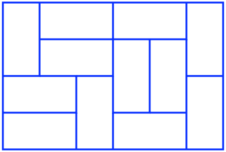
How many tilings are there?

The first really interesting result on counting tilings:



Theorem (Kasteleyn, Fisher-Temperley, 1961.)
The number of domino tilings of a $2m \times 2n$ rectangle is:

Any guesses?

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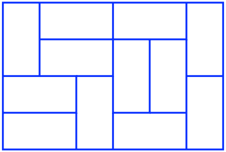

A hint:

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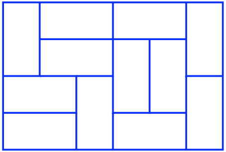
The number of domino tilings of a $2m \times 2n$ rectangle is:

$$4^{mn} \prod_{j=1}^{m} \prod_{k=1}^{n} \left(\cos^2 \frac{j\pi}{2m+1} + \cos^2 \frac{k\pi}{2n+1} \right).$$

This is an amazing formula; why is this number an integer?

For $m = 2$, $n = 3$:



$$
\begin{aligned}
& 4^6 \cdot (\cos^2 36^\circ + \cos^2 25.71\ldots^\circ) \cdot (\cos^2 72^\circ + \cos^2 25.71\ldots^\circ) \\
& \cdot(\cos^2 36^\circ + \cos^2 51.43\ldots^\circ) \cdot (\cos^2 72^\circ + \cos^2 51.43\ldots^\circ) \\
& \cdot(\cos^2 36^\circ + \cos^2 77.14\ldots^\circ) \cdot (\cos^2 72^\circ + \cos^2 77.14\ldots^\circ) \\
= \; & 4^6(1.466\ldots)(.907\ldots)(1.043\ldots)(.484\ldots)(.704\ldots)(.145\ldots) \\
= \; & 281
\end{aligned}
$$

- statistical mechanics: “dimers" in a square grid
- enumerative and algebraic combinatorics / linear algebra: determinants $\rightarrow$ eigenvalues/eigenvectors

Ok, but how big is this?

Theorem. (Kasteleyn, Fisher-Templerley, 1961.)
The number of domino tilings of a $2n \times 2n$ square is approximately

$$C^{4n^2}$$

where

$$C = e^{G/\pi} = \mathbf{1.338515152\ldots}$$

and

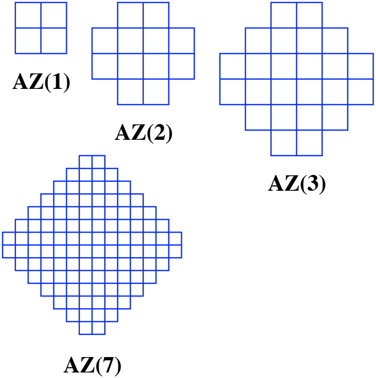
$$e = 1 + \frac{1}{1!} + \frac{1}{2!} \cdots = 2.718\ldots \qquad G = 1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} \cdots = 0.915\ldots$$

are Euler's and Catalan's constants.

Think like physicists:

If each cell chose its partner independently, we would have $1.3385\ldots$ degrees of freedom per cell.

Domino tilings of Aztec diamonds:



Question. Why “Aztec"?

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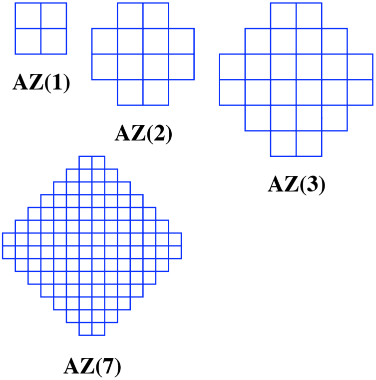

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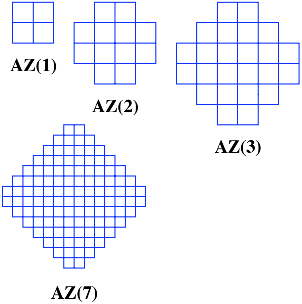
In honor of Mayan pyramids:

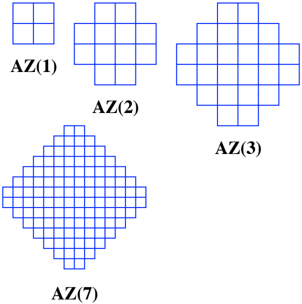


(**Mayan** Temple of Kukulkán, Chichén Itzá, 9th-12th Century)
Geographic confusion aside, they could be “Mayan diamonds".

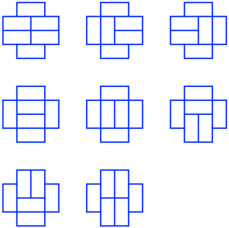
Theorem

“Aztec" diamonds are as Aztec as “Colombian" configurations are Colombian.

Domino tilings of Aztec diamonds.



AZ(2) has 8 tilings.



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$$2^{n(n+1)/2}.$$

An extremely simple answer! An extremely simple proof?

"We only understand something when we find it obvious."
There are ≥ 12 proofs so far; no obvious one yet.

If each cell chose its partner independently, we would have
$\sqrt[4]{2} = 1.1892\ldots$ degrees of freedom per cell.

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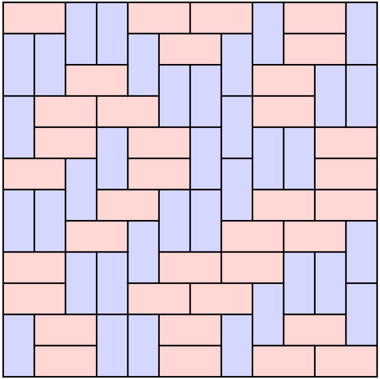
What does a typical tiling look like?

Squares and Aztec diamonds are really **very** different.

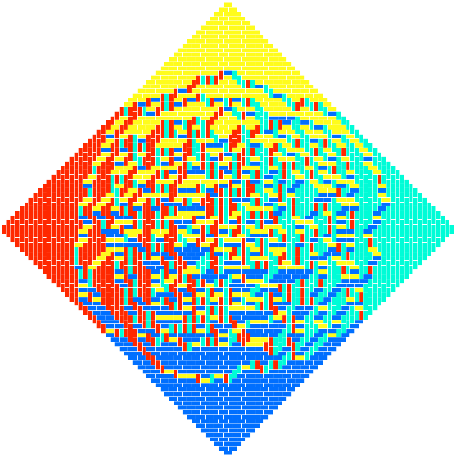
Here is a random tiling of a square:

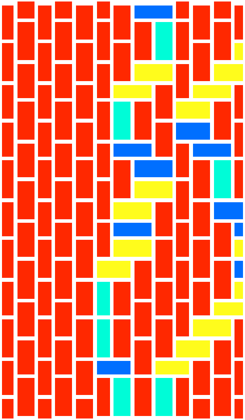


There is no obvious structure here.

Recall: 1.3385 . . . degrees of freedom per cell.

Compare it with a random tiling of an Aztec diamond:

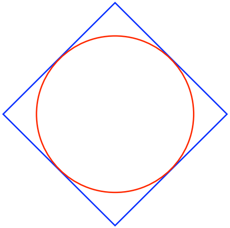




Completely regular near the corners, and chaotic in the middle!

More rigid: 1.1892 . . . degrees of freedom per cell.

Theorem. (Jockusch, Propp, Shor, 1995.)
For *n* “large", “almost all" tilings of the Aztec diamond AZ(*n*) exhibit a zone of regularity “very close" to the exterior of the arctic circle tangent to the four sides of the diamond.



The definitions of “large", “almost all", and “very close" are technical, but they explain what we see in the pictures.

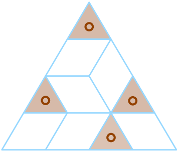
This is only one of many probabilistic results of this kind.

Tilings from geometry.

Suppose we wish to tile an equilateral triangle with rhombi:

It is imposible: $\#\Delta = \binom{n+1}{2}$ $\#\nabla = \binom{n}{2}$

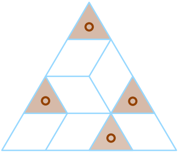
It may be possible if I cut out n holes Δ:



I wish to cut out n holes from the triangle of size n, so that the resulting “holey triangle" can be tiled.

Which n holes may I cut out?

Theorem. Colombian proposal to IMO Shortlist 2006
A holey triangle can be tiled with rhombi if and only if each triangle of side length k contains at most k holes.



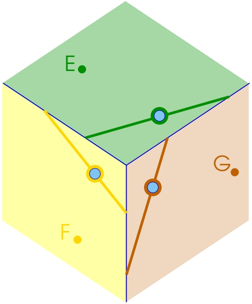
Where did this problem come from?

A **flag** $E_\bullet$ in space is a collection $E_0 \subset E_1 \subset E_2 \subset E_3 = \mathbb{R}^3$ of

a point E_0 in a line E_1 in a plane E_2 in space $E_3 = \mathbb{R}^3$

Goal: understand how flags intersect.

Example: three flags $E_\bullet, F_\bullet, G_\bullet$ in space.



Similarly in $\mathbb{R}^n$ a flag is:

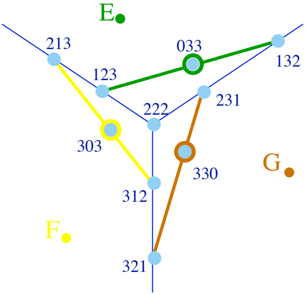
$$\emptyset \subset \text{point} \subset \text{line} \subset \text{plane} \subset \text{3-plane} \subset \text{4-plane} \subset \cdots \subset \mathbb{R}^n.$$

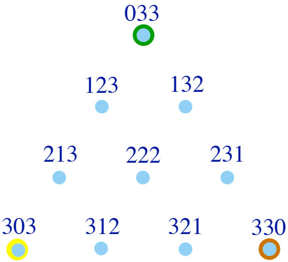
How do three general flags $E_\bullet, F_\bullet, G_\bullet$ intersect in $\mathbb{R}^3$?

To start, consider the 10 intersection points. They are:

$$E_i \cap F_j \cap G_k \qquad i + j + k = 6.$$

We write 231 for $E_2 \cap F_3 \cap G_1$.

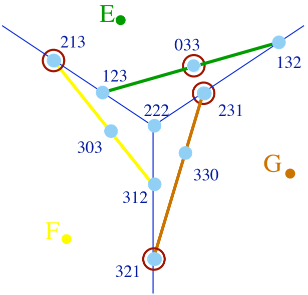


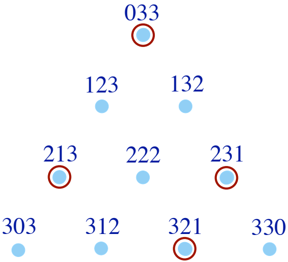


We obtain 10 points. Which 4-tuples are in **general position**?

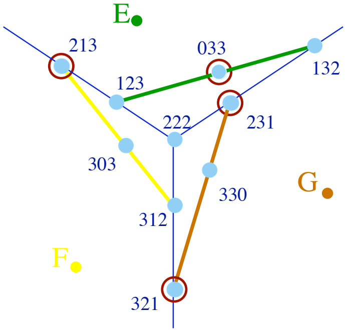
- No two are equal
- No three are collinear
- No four are coplanar.

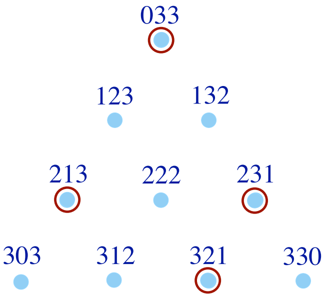
> **Question.** In this configuration of 10 points in $\mathbb{R}^3$, which subsets of 4 points are in general position?





Conjecture. (Billey–Vakil)
In this configuration of $\binom{n+1}{2}$ points in $\mathbb{R}^n$, an $(n+1)$-tuple of points is in general position if and only if each triangle of size k contains at most k points.





Theorem. (Ardila-Billey, 05)
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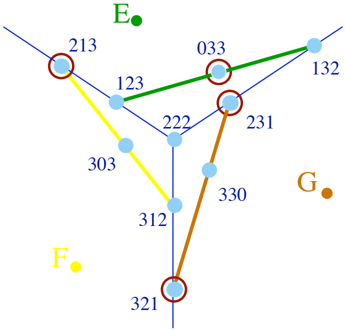
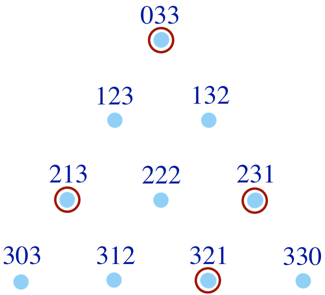
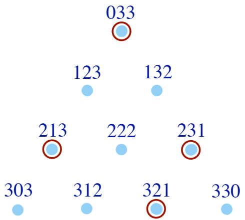


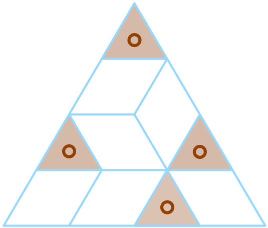


What about the tilings?



← Points in general position.

Lozenge tiling of an equilateral triangle →



The “holey triangle" tiling problem was crucial in our proof.

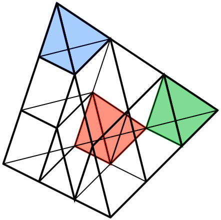
An open problem.

3 flags in $\mathbb{R}^n$ $\rightarrow$ tilings of a triangle of length n

4 flags in $\mathbb{R}^n$ $\rightarrow$ tilings of a tetrahedron of length n

$\vdots$ $\vdots$

Tiles: tetrahedra, triangular prisms, parallelepipeds.



Theorem. (Ardila, Billey, 2005)
In a tiling of the tetrahedron of length n,
(a) there are exactly n tetrahedra, and
(b) every tetrahedron of length k contains at most k tiles which are tetrahedra.

Conjecture. If n tetrahedra satisfy (a) and (b), the rest of the tetrahedron can be tiled with prisms and parallelepipeds.

(And similarly in any dimension.)

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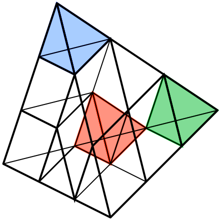

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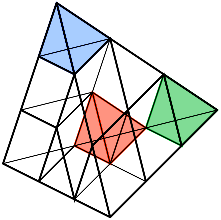

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Many thanks for your attention.

I will put these slides on my website.
To learn more:

Article: Tilings/Teselaciones/Pflasterungen (w/ Richard Stanley)
Lectures: 5 courses, $\geq$ 200 hours of videos in combinatorics:

`http://math.sfsu.edu/federico`
`http://youtube.com/user/federicoelmatematico`