counting lattice points in polytopes

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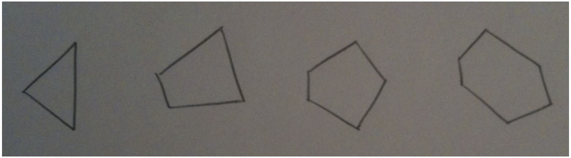
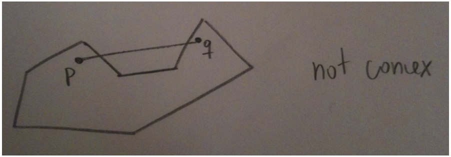
2-D: Polygons



Let's focus on *convex* polygons:

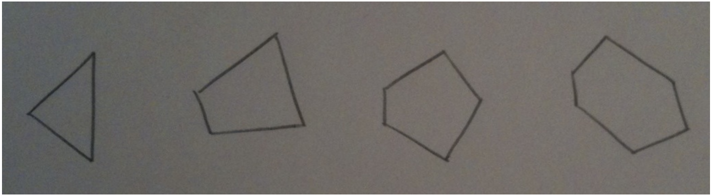


P is convex if:

If $p, q \in P$, then the whole line segment pq is in P.

2-D: Polygons

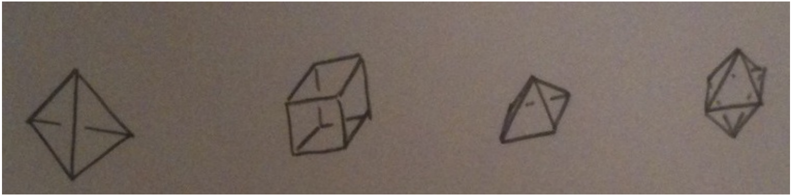
Combinatorially, polygons are very simple:



- One “combinatorial" type of *n*-gon for each *n*.
- One “regular" n-gon for each *n*.

3-D: Polyhedra

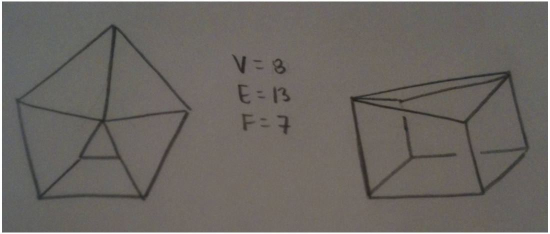
The combinatorial types in 3-D are much more complicated (and interesting)



Combinatorial type doesn't depend just on number of vertices.
Keep track of numbers V, E, F of vertices, edges, and faces.

3-D: Polyhedra

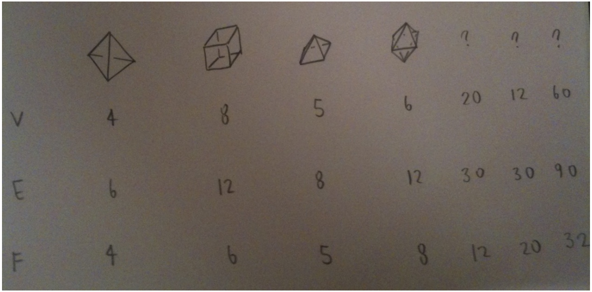
Keep track of numbers V, E, F of vertices, edges, and faces.
But even that is not enough! Two combinatorially different polytopes can have the same numbers (V, E, F)



Also, not every combinatorial type has a “regular" polytope.

Only regular polytopes:
tetrahedron, cube, octahedron, dodecahedron, icosahedron.

3-D: Polyhedra



Theorem. (Euler 1752) $V - E + F = 2$.

Klee: “first landmark in the theory of polytopes"
Alexandroff-Hopf: “first important event in topology"

3-D: Polyhedra

Question.
Given numbers V, E, F, can you construct a polytope with V vertices, E edges, and F faces?

Theorem. (Steinitz, 1906)
There exists a polytope with V vertices, E edges, and F faces if and only if

$$V - E + F = 2, \qquad V \leq 2F - 4, \qquad F \leq 2V - 4.$$

4-D: Polychora

In 4-D, things are even more complicated (and interesting).

Euler's theorem?
$V - E + F - S = 0$.

Steinitz's theorem?
Not yet. But Ziegler et. al. have made significant progress.

Regular polychora?
simplex, cube, crosspolytope, 24-cell, 120-cell, 600-cell
Discovered by Ludwig Schläfli and by Alicia Boole Stott.
(She coined the term "polytope".)

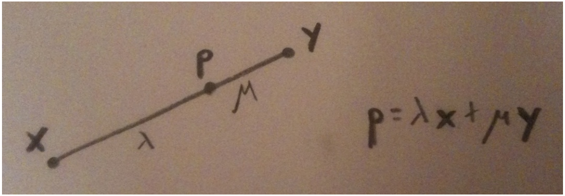
But what are we even talking about? **What is a polytope?**

n-D: Polytopes

What is a polytope? (First answer.)

In 1-D: “polytope" = segment

$$\begin{aligned}\overline{\mathbf{xy}} &= \text{“convex hull" of } \mathbf{x} \text{ and } \mathbf{y}\\ &= \{\lambda\mathbf{x} + \mu\mathbf{y} \;:\; \lambda, \mu \geq 0, \lambda + \mu = 1\}\end{aligned}$$

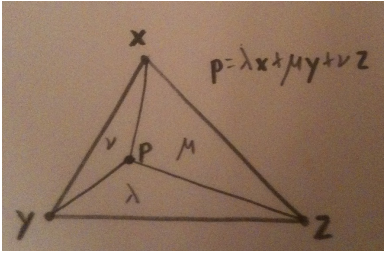


n-D: Polytopes

What is a polytope? (First answer.)

In 2-D:

$$\begin{aligned}\triangle\mathbf{xyz} &= \text{“convex hull" of } \mathbf{x}, \mathbf{y}, \text{ and } \mathbf{z} \\ &= \mathbf{conv}(\mathbf{x},\mathbf{y},\mathbf{z}) \\ &= \{\lambda\mathbf{x}+\mu\mathbf{y}+\nu\mathbf{z} \,:\, \lambda,\mu,\nu \geq 0, \lambda+\mu+\nu = 1\}\end{aligned}$$



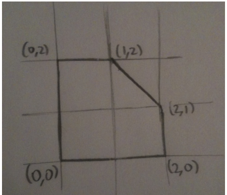
n-D: Polytopes

What is a polytope? (First answer.)

Definition. A polytope is the convex hull of m points in $\mathbb{R}^d$:

$$P = \textbf{conv}(\mathbf{v}_1, \ldots, \mathbf{v}_m) := \left\{ \sum_{i=1}^{m} \lambda_i \mathbf{v}_i \ : \ \lambda_i \geq 0, \sum_{i=1}^{m} \lambda_i = 1 \right\}$$

Trouble: Hard to tell whether a point is in the polytope or not.



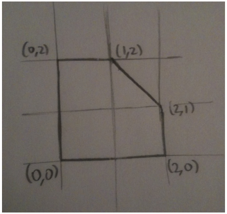
n-D: Polytopes

What is a polytope? (Second answer.)

Definition. A polytope is the solution to a system of linear inequalities in $\mathbb{R}^d$:

$$P = \left\{ \mathbf{x} \in \mathbb{R}^d \ : \ A\mathbf{x} \leq \mathbf{b} \right\}$$

(Trouble: Hard to tell what are the vertices.)



n-D: Polytopes

What is a polytope? (Both answers are the same.)

Theorem. A subset of $\mathbb{R}^d$ is the convex hull of a finite number of points if and only if it is the bounded set of solutions to a system of linear inequalities.

Note.
It is tricky, but possible, to go from the **V-description** to the **H-description** of a polytope. The software `polymake` does it.

Examples of polytopes

1. Simplices

(point, segment, triangle, tetrahedron, ...)

Let $\mathbf{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}^d$. (1 in ith position)

The standard $(d-1)$-simplex is

$$\begin{aligned} \Delta_{d-1} &:= \mathbf{conv}(\mathbf{e}_1, \ldots, \mathbf{e}_d) \\ &= \left\{ \mathbf{x} \in \mathbb{R}^d : x_i \geq 0, \sum_{i=1}^{d} x_i = 1 \right\} \end{aligned}$$

Examples of polytopes

2. Cubes

(point, segment, square, cube, . . .)

The *standard d-cube* is

$$\begin{aligned} \square_d \quad &:= \quad \mathbf{conv}(\, \mathbf{b} \;:\; \text{all } b_i \text{ equal 0 or 1}) \\ &= \quad \left\{ \mathbf{x} \in \mathbb{R}^d \;:\; 0 \leq x_i \leq 1 \right\} \end{aligned}$$

Examples of polytopes

3. Crosspolytopes

(point, segment, square, octahedron, . . .)

The *d-crosspolytope* is

$$\begin{aligned}\Diamond_d &:= \mathbf{conv}(-\mathbf{e}_1, \mathbf{e}_1, \ldots, -\mathbf{e}_d, \mathbf{e}_d) \\ &= \left\{ \mathbf{x} \in \mathbb{R}^d : \sum_{i=1}^{d} a_i x_i \leq 1 \text{ whenever all } a_i \text{ are } -1 \text{ or } 1 \right\}\end{aligned}$$

Two very nice facts about polytopes. (Schläfli, 1850)

- Euler's theorem:
  If f_i is the number of i-dimensional faces, then

$$f_1 - f_2 + \cdots \pm f_{d-1} = \begin{cases} 0 & \text{if } d \text{ is even,} \\ 2 & \text{if } d \text{ is odd.} \end{cases}$$

  (Roots of algebraic topology.)

- Classification of regular polytopes:
  The only regular polytopes are:
  - the d-simplices (all d),
  - the d-cubes (all d),
  - the d-crosspolytopes (all d),
  - the icosahedron and dodecahedron ($d = 3$),
  - the 24-cell, the 120-cell, and the 600-cell ($d = 4$).

  (Roots of Coxeter group theory.)

Corte de comerciales.

For more information about

- polytopes
- Coxeter groups,
- matroids, and
- combinatorial commutative algebra,

please visit

`http://math.sfsu.edu/federico/`

where you will find links to the (150+) lecture videos and notes of my courses at San Francisco State University and the Universidad de Los Andes.

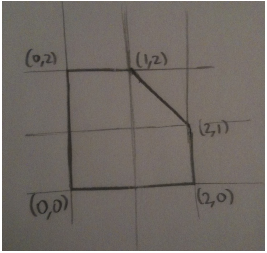
(Also exercises, discussion forum, research projects, etc.)

(San Francisco State University – Colombia Combinatorics Initiative)

Polygons

Question. How does the combinatorialist measure a polytope?
Answer. By counting! (Counting what?)

Continuous measure: area
Discrete measure: number of lattice points



area: $3\frac{1}{2}$ lattice points: 8 total, 1 interior.

Polygons

Continuous measure: area

Discrete measure: number of lattice points

From discrete to continuous:

Theorem. (Pick, 1899)

Let P be a polygon with integer vertices. If

I = number of interior points of P and

B = number of boundary points of P, then

$$\text{Area}(P) = I + \frac{B}{2} - 1$$

In the example,

$$\text{Area}(P) = \frac{7}{2}, \quad I = 1, \quad B = 7.$$

Polytopes

To extend to n dimensions, we need to count more things.

Continuous measure: volume = $\int_P dV$
Discrete measure: number of lattice points
A richer discrete measure:

Let $L_P(n) =$ number of lattice points in nP.
Let $L_{P^o}(n) =$ number of interior lattice points in nP.

In example,

$$L_P(n) = \frac{7}{2}n^2 + \frac{7}{2}n + 1, \qquad L_{P^o}(n) = \frac{7}{2}n^2 - \frac{7}{2}n + 1.$$

Examples

2. Cube

In 3-D, $L_{\square_3}(n) = (n+1)^3$ (a cubical grid of size $n+1$)

$L_{\square_3^o}(n) = (n-1)^3$ (a cubical grid of size $n-1$)

Examples

2. Cube

In dimension 3, $L_{\square_3}(n) = (n+1)^3, \quad L_{\square_3^o}(n) = (n-1)^3$.

In dimension d, we need to count lattice points in

$$n\square_d = \left\{ \mathbf{x} \in \mathbb{R}^d \,:\, 0 \leq x_i \leq n \right\}.$$

Lattice points:

$(y_1, \ldots, y_d) \in \mathbb{Z}^d$ with $0 \leq y_i \leq n$. $\quad$ ($n+1$ options for each y_i)

Interior lattice points:

$(y_1, \ldots, y_d) \in \mathbb{Z}^d$ with $0 < y_i < n$. $\quad$ ($n-1$ options for each y_i)

$$L_{\square_d}(n) = (n+1)^d, \qquad L_{\square_d^o}(n) = (n-1)^d.$$

Examples

1. Simplex

Count points in $$n\Delta_{d-1} = \left\{ \mathbf{x} \in \mathbb{R}^d \,:\, x_i \geq 0, \sum_{i=1}^{d} x_i = n \right\}.$$

Interior points: $(y_1, \ldots, y_d) \in \mathbb{Z}^d$ with $y_i > 0, \sum_{i=1}^{d} y_i = n$.

$$L_{\Delta_d^o}(n) = \binom{n-1}{d-1}.$$

Lattice points: $(y_1, \ldots, y_d) \in \mathbb{Z}^d$ with $y_i \geq 0, \sum_{i=1}^{d} y_i = n$.
$z_i = y_i + 1 \leftrightarrow (z_1, \ldots, z_d) \in \mathbb{Z}^d$ with $z_i > 0, \sum_{i=1}^{d} z_i = n + d$.

$$L_{\Delta_d}(n) = \binom{n+d-1}{d-1}.$$

Examples

3. Crosspolytope: Skip.

4. The "coin polytope"

Let $f(N)$ = number of ways to make change for N cents using (an unlimited supply of) quarters, dimes, nickels, and pennies.

Notice: $f(N)$ is the number of lattice points in the polytope

$$\mathrm{Coin}(N) = \{(q, d, n, p) \in \mathbb{R}^4 : q, d, n, p \geq 0,\ 25q + 10d + 5n + p = N\}$$

Now, $\mathrm{Coin}(N) = N\mathrm{Coin}(1)$, so

$$L_{\mathrm{Coin}(1)}(N) = f(N), \qquad L_{\mathrm{Coin}(1)^o}(N) = f(N - 41).$$

Warning: Coin(1) does not have integer vertices.

Ehrhart's theorem

Theorem. (Ehrhart, 1962)

Let P be a d-polytope with integer vertices in $\mathbb{R}^d$. Then

$$L_P(n) = c_d n^d + c_{d-1} n^{d-1} + \cdots + c_1 n + c_0$$

is a polynomial in n of degree d. Also,

$$c_d = \mathrm{Vol}(P), \qquad c_{d-1} = \text{“Surface Vol”}(P), \qquad c_0 = 1.$$

So (discrete) counting gives us the (continuous) volume of P.

This is called the Ehrhart polynomial of P.

Ehrhart's theorem

Theorem. (Ehrhart, 1962) For a d-polytope P,

$$L_P(n) = c_d n^d + c_{d-1} n^{d-1} + \cdots + c_1 n + c_0$$

is the Ehrhart polynomial of P.

Define the Ehrhart series of P to be

$$Ehr_P(z) = \sum_{n \geq 0} L_P(n) z^n = L_P(0)z^0 + L_P(1)z^1 + L_P(2)z^2 + \cdots$$

In our first example,

$$Ehr_P(z) = \sum_{n \geq 0} \left(\frac{7}{2} n^2 + \frac{7}{2} n + 1 \right) z^n = \cdots = \frac{1 + 5z + z^2}{(1-z)^3}$$

for $|z| < 1$.

Examples

1. Simplex. We computed the Ehrhart polynomial:

$$L_{\Delta_d}(n) = \binom{n+d-1}{d-1} = \binom{n+d-1}{n} = \frac{(n+d-1)(n+d-2)\ldots(d+1)d}{d!}.$$

Notice that:

$$\binom{-d}{n} = \frac{(-d)(-d-1)\cdots(-n-d+2)(-n-d+1)}{d!} = (-1)^n\binom{n+d-1}{d-1}$$

so

$$Ehr_{\Delta_d}(z) = \sum_{n\geq 0} L_{\Delta_d}(n)z^n = \sum_{n\geq 0}(-1)^n\binom{-d}{n}z^n = (1-z)^{-d}.$$

In conclusion,

$$Ehr_{\Delta_d}(z) = \frac{1}{(1-z)^d}$$

Examples

2. Cube. We computed the Ehrhart polynomial:

$L_{\square_d}(n) = (n+1)^d$ so $Ehr_{\square_d}(z) = \sum_{n\geq 0}(n+1)^d z^n.$

$Ehr_{\square_0}(z) = \frac{1}{1-z}$, $Ehr_{\square_1}(z) = \frac{1}{(1-z)^2}$, $Ehr_{\square_2}(z) = \frac{1+z}{(1-z)^3}$

$Ehr_{\square_3}(z) = \frac{1+4z+z^2}{(1-z)^4}$, $Ehr_{\square_4}(z) = \frac{1+11z+11z^2+z^3}{(1-z)^5}, \ldots$

To compute these use $Ehr_{\square_{d+1}}(z) = Ehr_{\square_d}(z) + z\frac{d}{dz}Ehr_{\square_d}(z)$

We are led to guess that

$$Ehr_{\square_d}(z) = \frac{a_0 z^0 + a_1 z^1 + \cdots + a_d z^d}{(1-z)^{d+1}}$$

where a_i is a positive integer. What does it count?

Examples

2. Cube. The Ehrhart series of the d-cube is:

$$Ehr_{\square_d}(z) = \sum_{n \geq 0} (n+1)^d z^n = \frac{a_0 z^0 + a_1 z^1 + \cdots + z_d z^d}{(1-z)^{d+1}}$$

Theorem. (Euler 1755 / Carlitz, 1953) The number a_i equals the number of permutations of $[n]$ having exactly i descents.

Example: The permutations of $\{1, 2, 3\}$ and their descents:

$$123, 132, 213, 231, 312, 321$$

So

$$Ehr_{\square_3}(z) = \frac{1 + 4z + 1z^2}{(1-z)^3}.$$

Examples

3. Crosspolytope The Ehrhart series of the d-crosspolytope is:

$$Ehr_{\Diamond_d}(z) = \frac{(1+z)^d}{(1-z)^{d+1}}.$$

Skip.

4. Coin polytope The Ehrhart series of the coin polytope is:

$$Ehr_{\text{Coin}}(z) = (1 + z^1 + z^{1\cdot 2} + z^{1\cdot 3} + \cdots)(1 + z^5 + z^{5\cdot 2} + z^{5\cdot 3} + \cdots)$$
$$(1 + z^{10} + z^{10\cdot 2} + z^{10\cdot 3} + \cdots)(1 + z^{25} + z^{25\cdot 2} + z^{25\cdot 3} + \cdots)$$

so

$$Ehr_{\text{Coin}}(z) = \frac{1}{(1-z)(1-z^5)(1-z^{10})(1-z^{25})}.$$

One of these is not like the others. Why?

Stanley's theorem

Theorem. (Stanley 1980) For any d-polytope with integer vertices, the Ehrhart series is of the form

$$Ehr_P(z) = \frac{a_0 z^0 + a_1 z^1 + \cdots + a_d z^d}{(1-z)^{d+1}}$$

where $a_0, \ldots, a_d$ are non-negative integers.

- If the vertices are rational, then for some integers $n_1, \ldots, n_{d+1} > 0$:

$$Ehr_P(z) = \frac{a_0 z^0 + a_1 z^1 + \cdots + a_d z^d}{(1-z^{n_1}) \cdots (1-z^{n_{d+1}})}$$

- If the vertices are irrational, nobody knows.

Strategy of proof: Prove it for simplices, then "triangulate".

Ehrhart reciprocity

If we plug $n \in \mathbb{N}$ into the Ehrhart polynomial, we get

$$L_P(n) = \text{ number of lattice points in } nP$$

A strange idea: What if we plug in a negative integer $-n$?

$$L_P(-n) = \text{ ??}$$

Something amazing happens:

Theorem. (Macdonald 1971) For any d-polytope with integer vertices,

$$L_P(-n) = (-1)^d L_{P^\circ}(n).$$

We get the number of interior points in nP!

Ehrhart reciprocity v2

Put differently,

Theorem. (Macdonald 1971) If the Ehrhart polynomial of P is

$$L_P(n) = c_d n^d + c_{d-1} n^{d-1} + \cdots + c_1 n + c_0$$

then the interior Ehrhart polynomial is

$$L_{P^o}(n) = c_d n^d - c_{d-1} n^{d-1} + \cdots \pm c_1 n \mp c_0.$$

For instance, recall that in our example:

$$L_P(n) = \frac{7}{2}n^2 + \frac{7}{2}n + 1, \qquad L_{P^o}(n) = \frac{7}{2}n^2 - \frac{7}{2}n + 1.$$

Back to Pick's theorem

Theorem. (Pick, 1899)
Let P be a polygon with integer vertices. If
I = number of interior points of P and
B = number of boundary points of P, then

$$\mathrm{Area}(P) = I + \frac{B}{2} - 1$$

Proof: We have

$$L_P(n) = an^2 + bn + c, \qquad L_{P^\circ}(n) = an^2 - bn + c.$$

Therefore

$$I = a - b + c, \quad B = 2b \qquad \longrightarrow \qquad I + \tfrac{B}{2} - 1 = a + c - 1.$$

But we saw that $a = \mathrm{Area}(P)$ and $c = 1$. $\square$

Thank you very much.

Muchas gracias.

Corte de comerciales.

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