Hidden patterns: the shape of multiplication

Math Encounters
Museum of Mathematics
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gracias, ny familia

León Armeni 7A

Pre-algebra 4-9-2018

Federico Ardila

Federico Ardila is an internationally recognized Colombian-American mathematician who specializes in combinatorics and polytopes. Though he is still young, he has made a name for himself as a dedicated professor and talented therotician. It might come as a surprise that Ardila grew up indifferent to academics, and more interested in becoming a professional soccer player than a professor -- but when he was introduced to mathematics, his life changed.

gracias, team

1. MULTIPLICATION

Question: (For you!) (Please don't scream out your answer yet.)

$$2 \times 3 \times 4 \times 5 = ?$$

What answer did you get?
How did you do it?
How did you feel doing it?

Talk to your neighbor.

Talk to your neighbor.

$$2 \times 3 \times 4 \times 5 = 120$$

Different processes give the same answer!!!

Who is suprised? Who is not surprised?

Why do we get the same answer?!?! Today's topic.

Two simple laws with complicated names:

(or: why we get the same answer)

Commutative Law: $a \times b = b \times a$

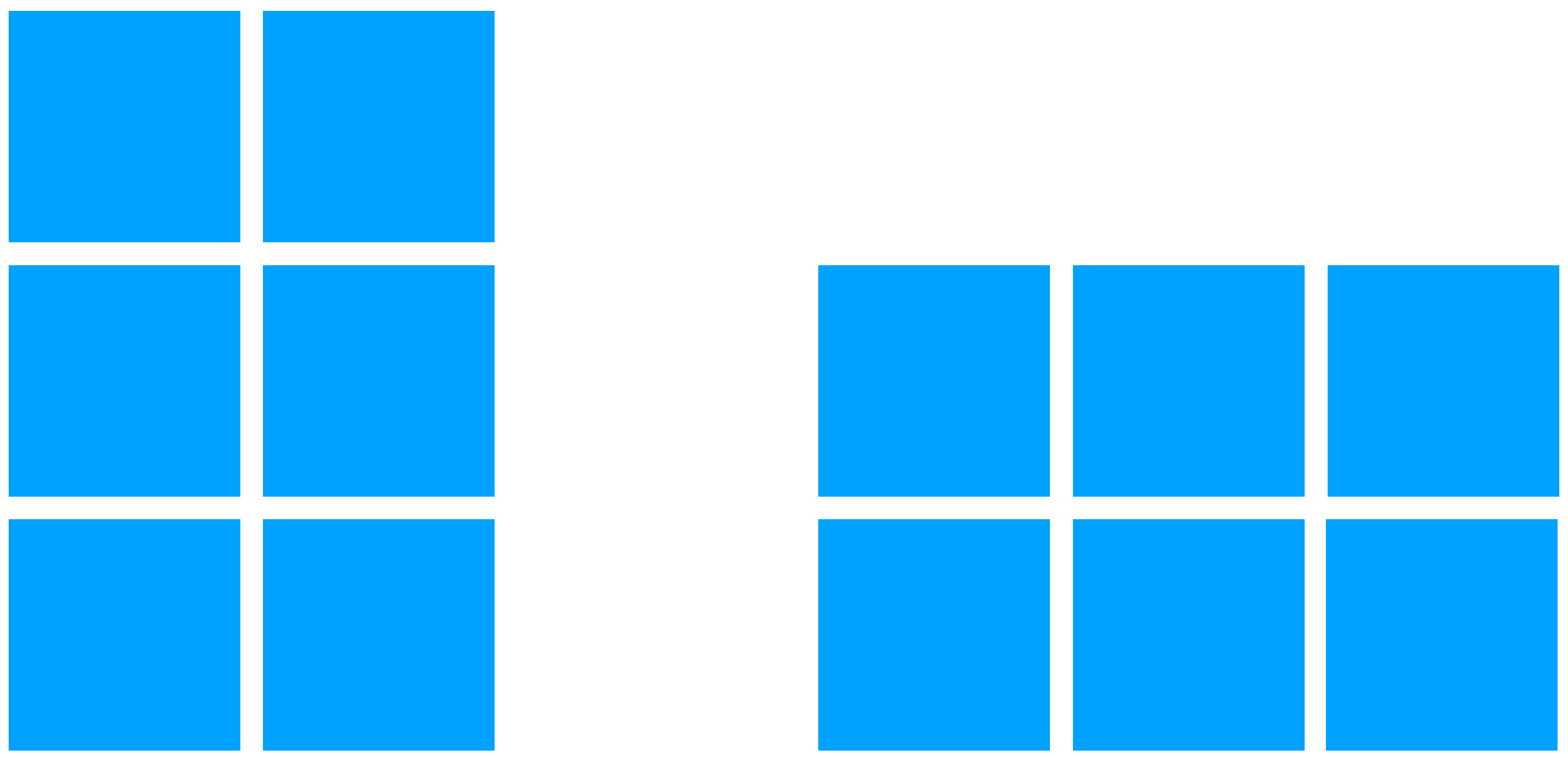
Associative Law: $(a \times b) \times c = a \times (b \times c)$

"The order of operations doesn't affect the result."

2. ORDER DOESN'T MATTER

Order doesn't matter: $a \times b = b \times a$

Why?!?! One illustration:



2×3 3×2

Is this a proof?

Not to a mathematician.

Is $202 \times 117 = 117 \times 202$?

We need to think more slowly.

Order doesn’t matter: a x b = b x a

Why? Terry Tao’s *Analysis 1*:

Lemma 2.3.2 (Multiplication is commutative). *Let n, m be natural numbers. Then $n \times m = m \times n$.*

Proof. See Exercise 2.3.1. $\square$

We learn **a lot** from thinking slowly about what we think we understand.

– We build solid foundations.

– We discover cool things we had missed.

Order doesn't matter: $a \times b = b \times a$

Why is $2 \times 3 \times 4 \times 5 = 5 \times 2 \times 4 \times 3$?

Why is $abcd = dacb$?

The commutative law is about 2 factors, not 4. Go slow!

$$abcd = abdc = adbc = dabc = dacb$$

Ok cool! But what about all the other orders of a,b,c,d?

Why do they **all** give the same answer?

Order doesn't matter: $ab = ba$

Questions: (For you!)

1. What are **all** the possible orders for multiplying a,b,c?

2. Why do they **all** give the same answer? Can you prove it?

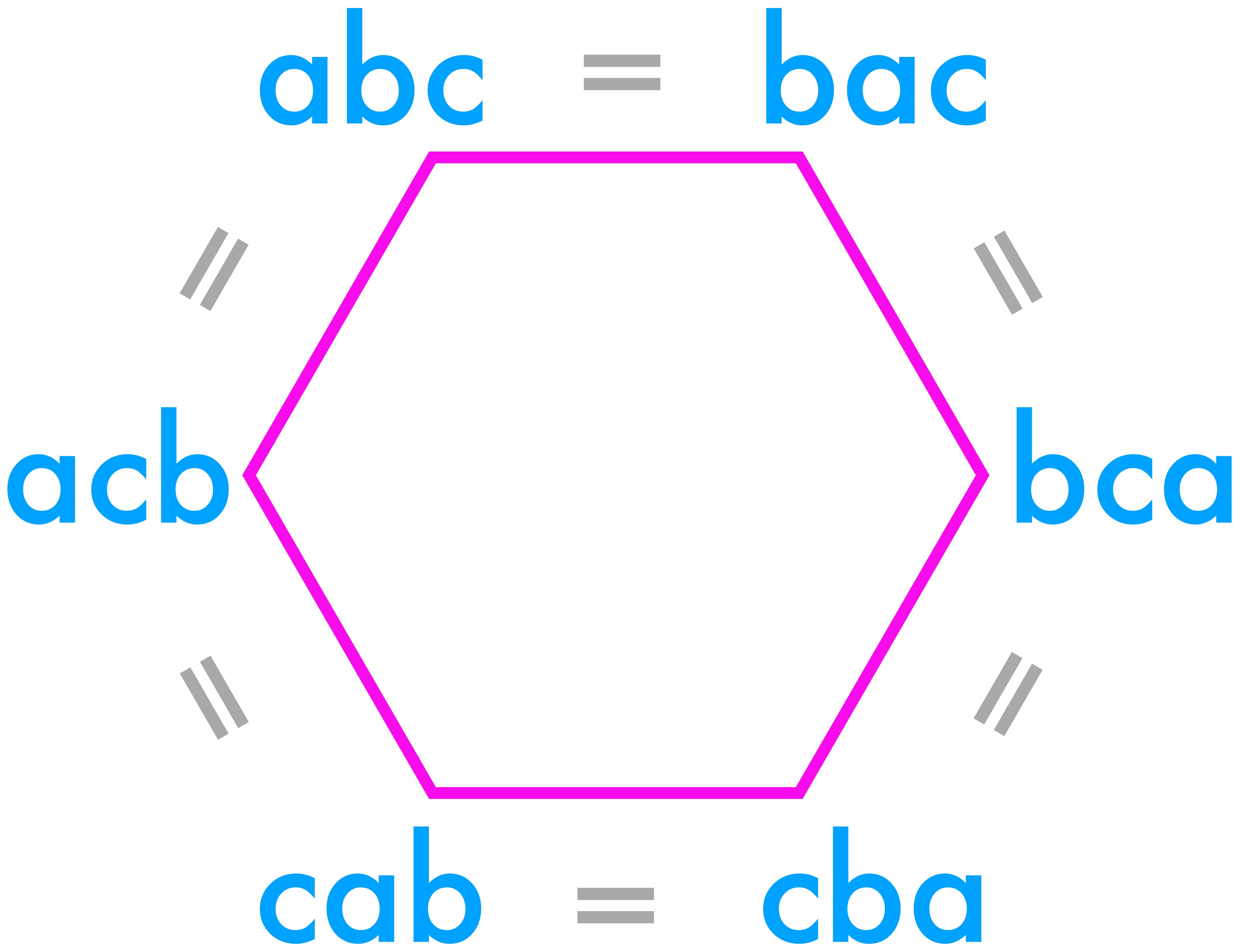
Talk to your neighbor.

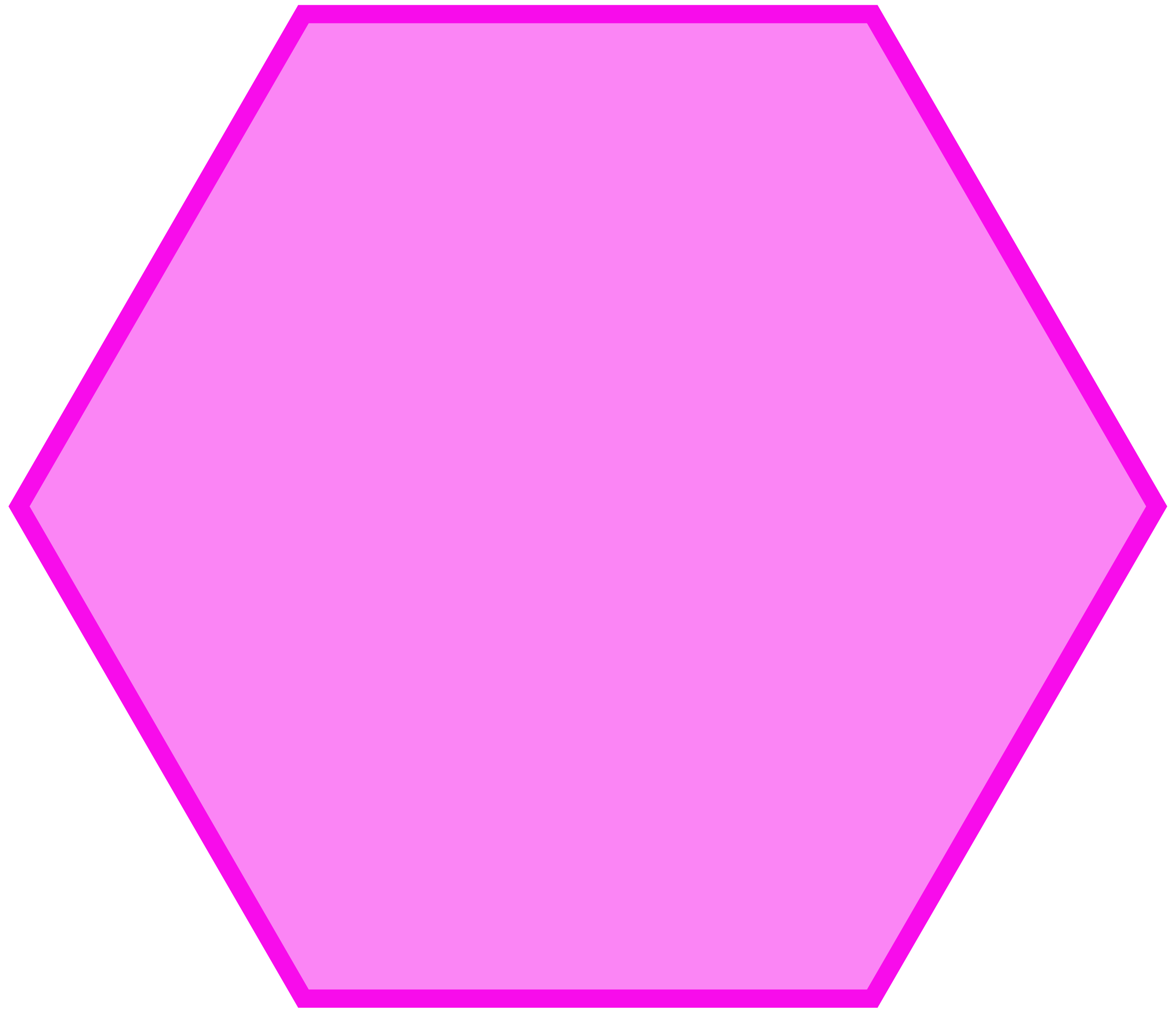
Talk to your neighbor.

Youth Speaks
Life is Living
Oakland

Order doesn't matter:

abc, acb, bac, bca, cab, cba are equal:





The shape of 3-commutativity.

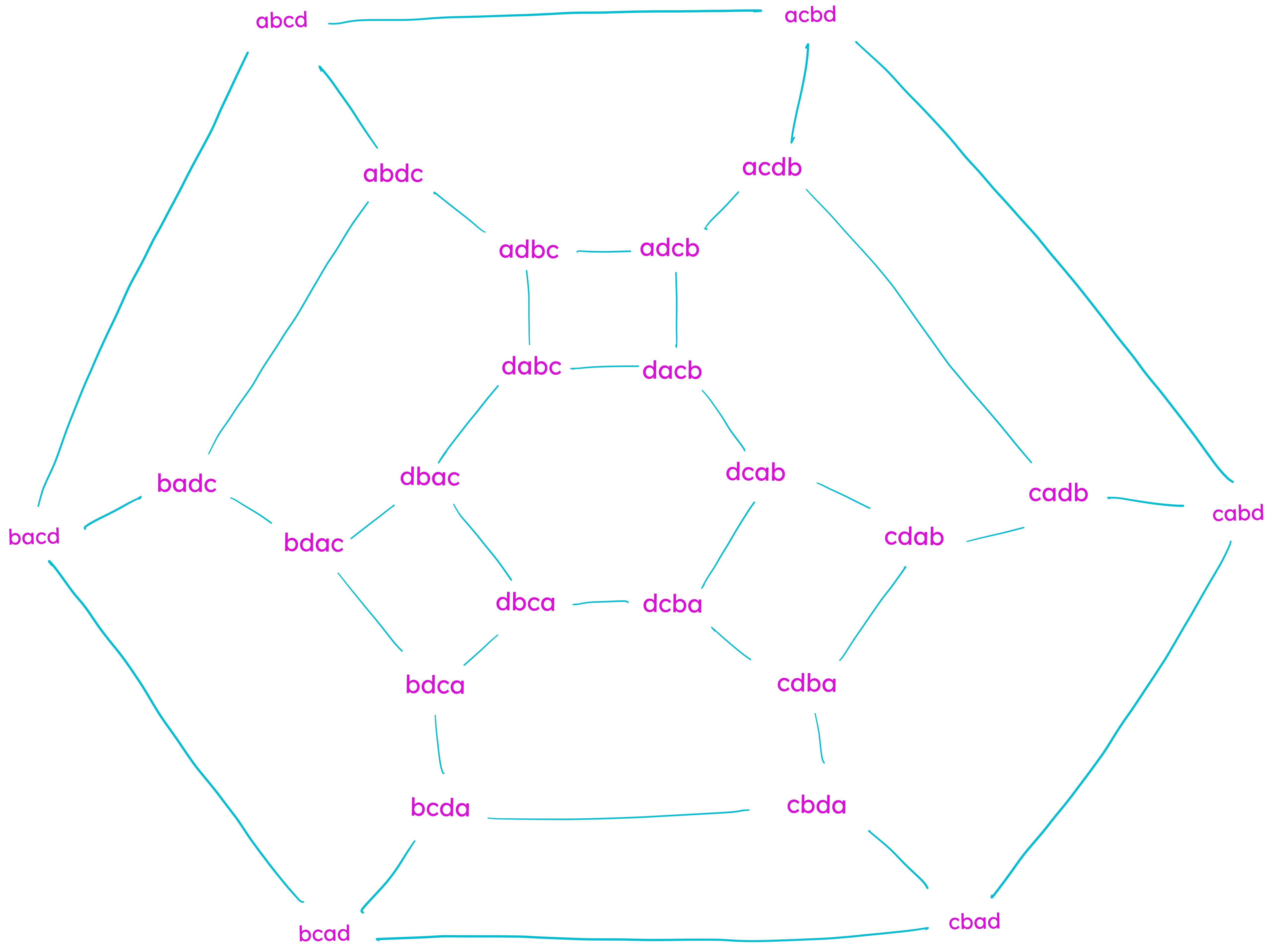
Order doesn't matter:

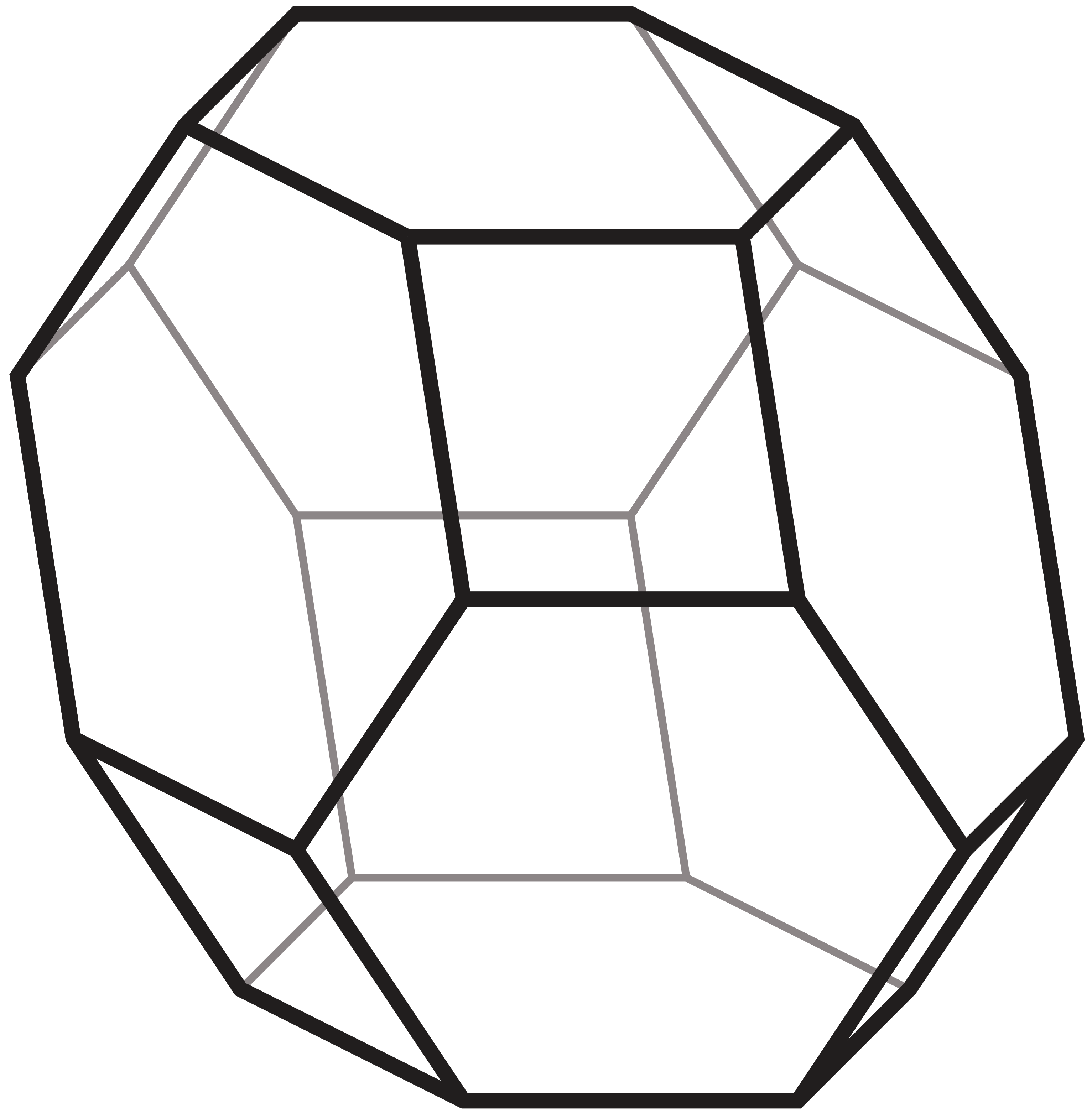
All 24 products abcd, ..., dcba are equal.

Let's apply the same method!

Order doesn't matter:

All 24 products abcd, ..., dcba are equal:

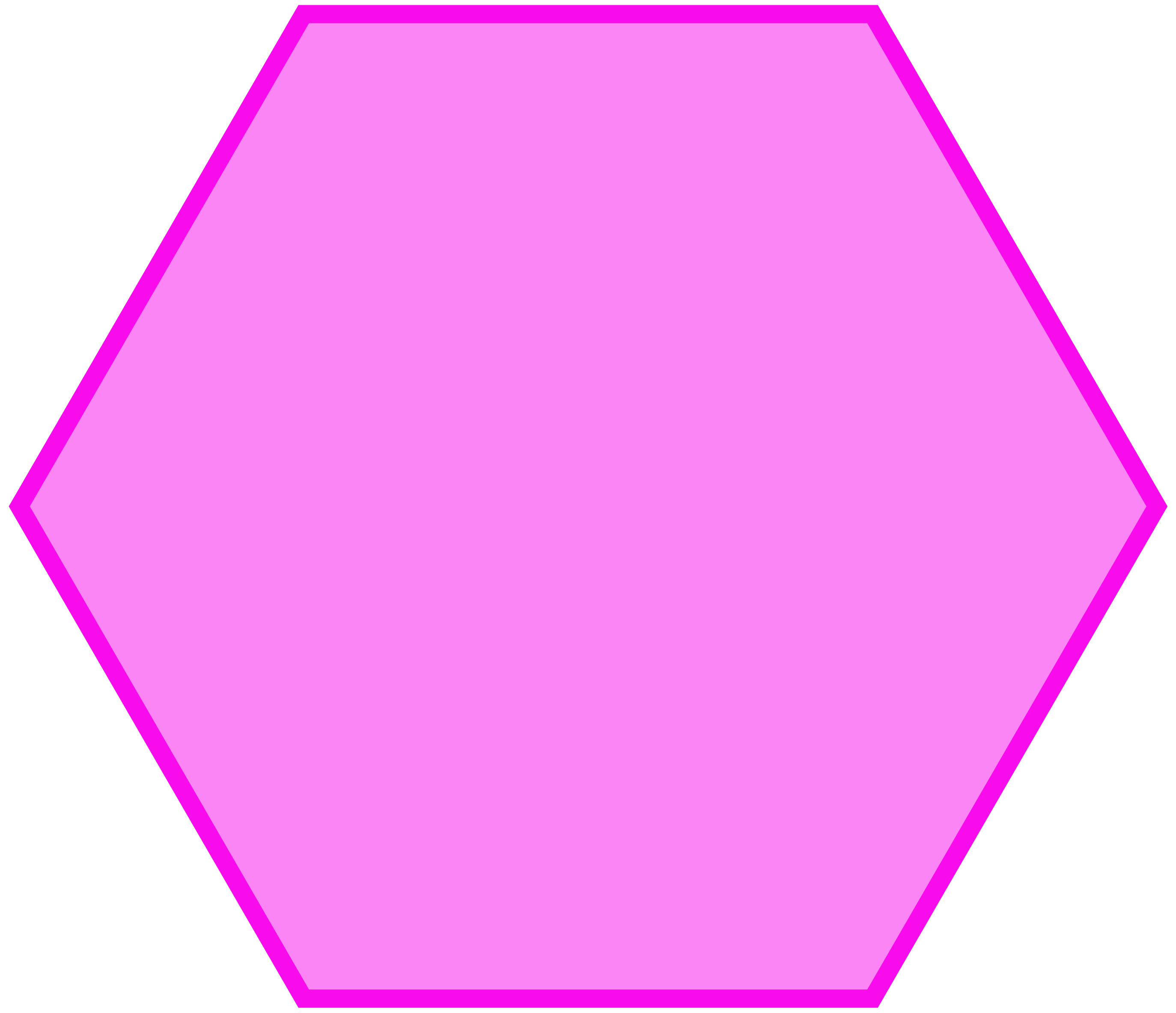




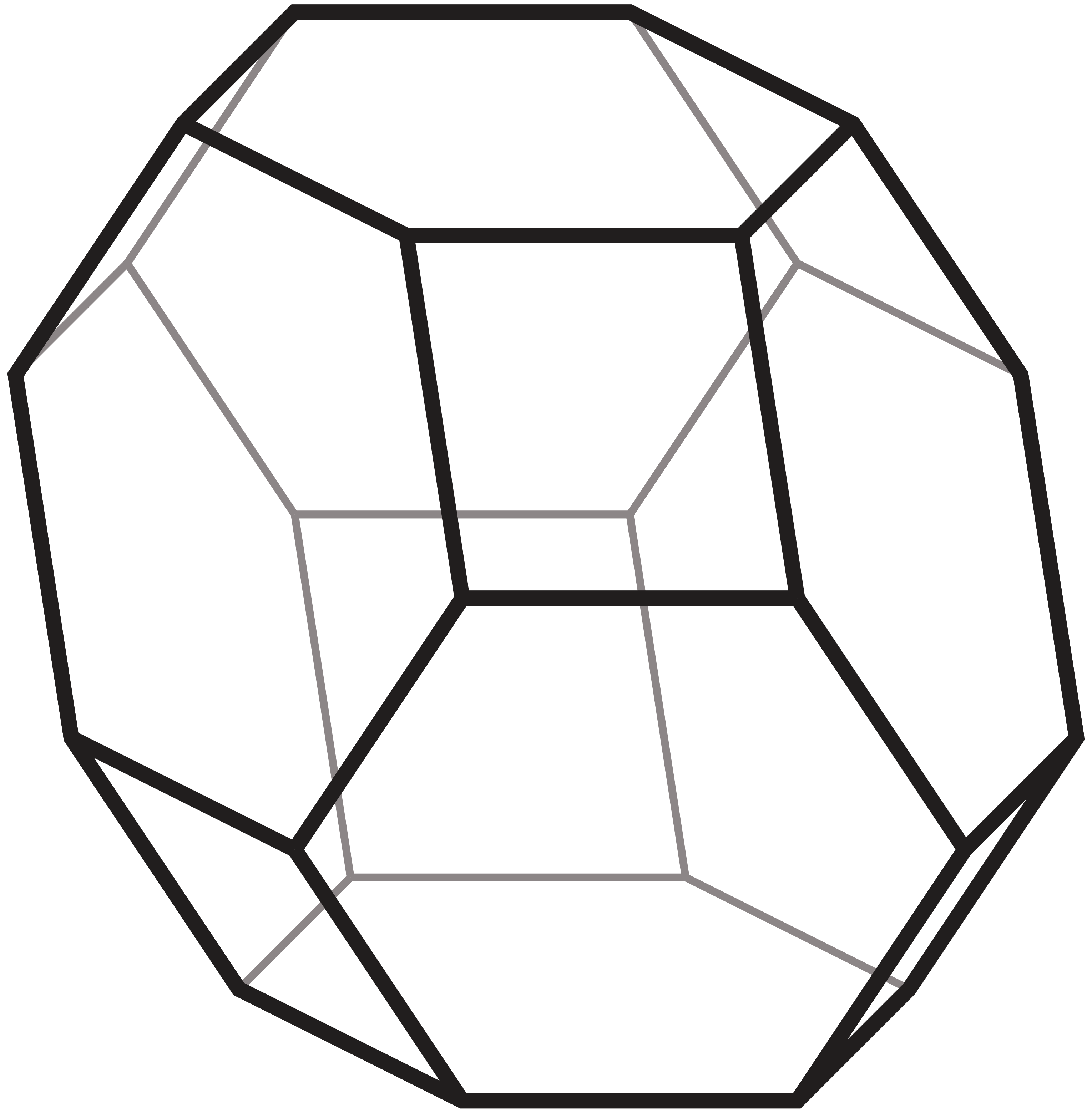
The shape of 4-commutativity.

3. THE PERMUTAHEDRON

The permutahedron



The shape of 3-commutativity.



The shape of 4-commutativity.

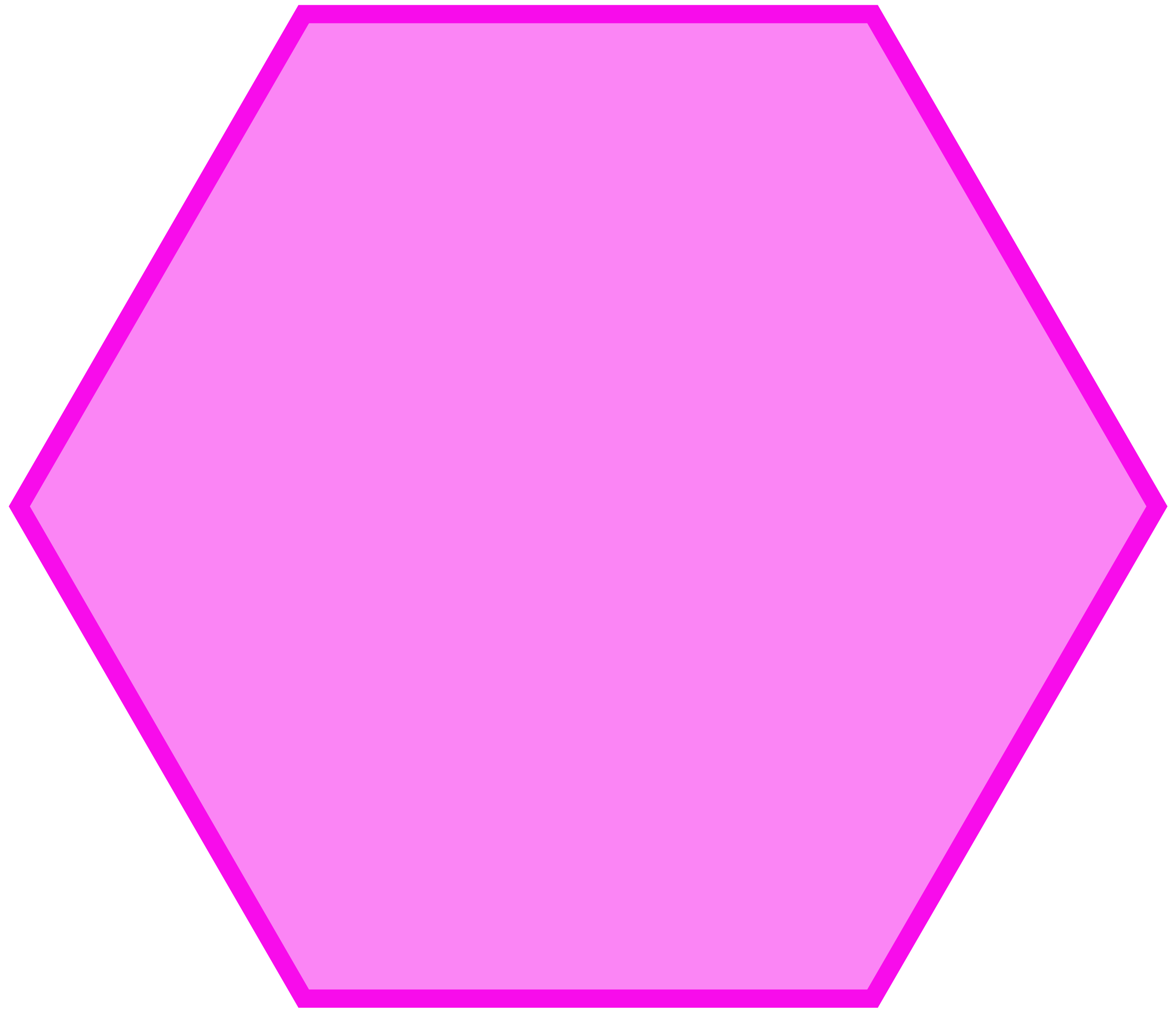
The permutahedron

One way to build it: (Schoute 1914)

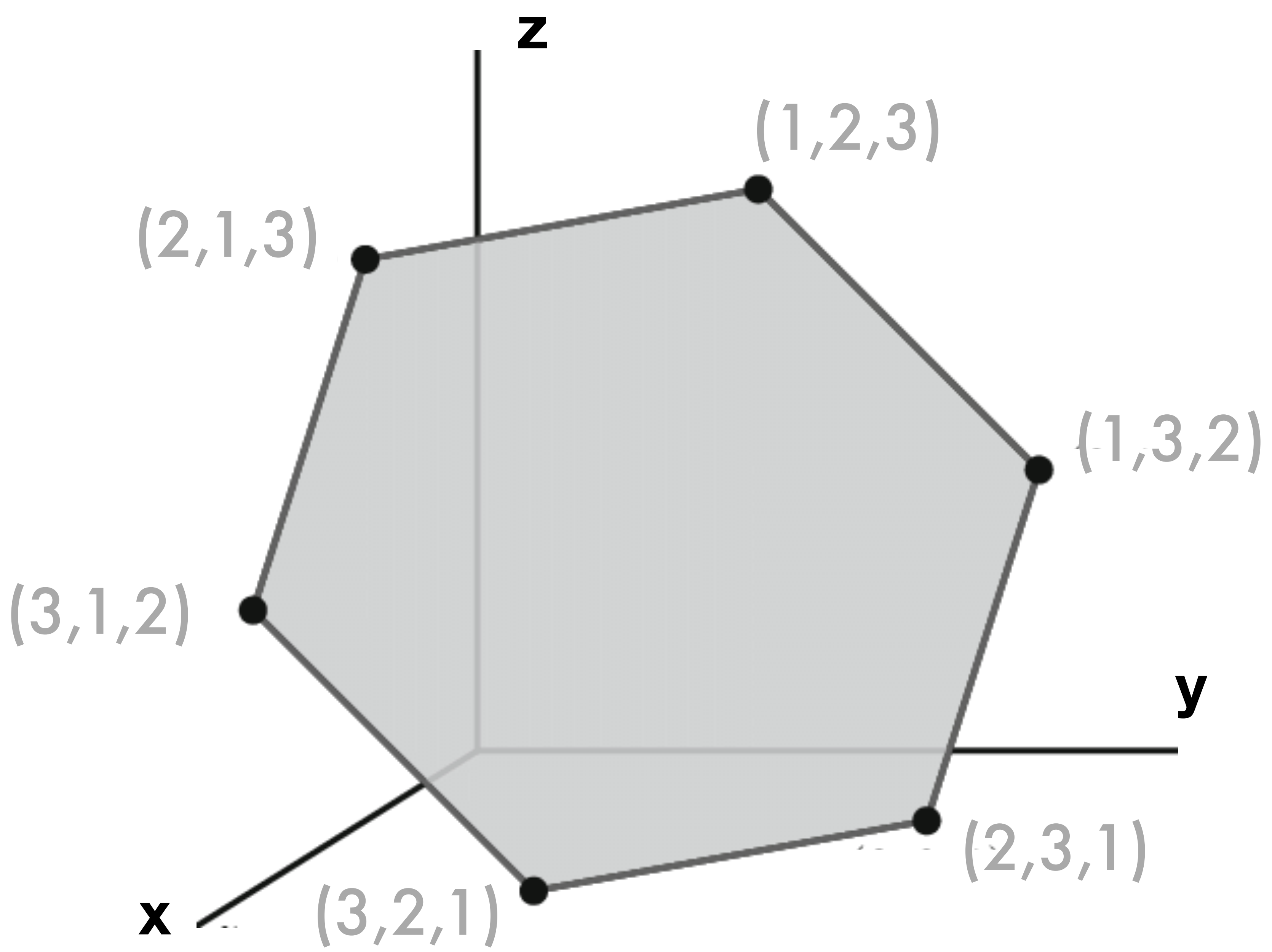
Gift wrap these 6 points in 3-D space:

(1,2,3) (1,3,2) (2,1,3) (2,3,1) (3,1,2) (3,2,1)

and you'll get the 2-permutahedron.



The shape of 3-commutativity.



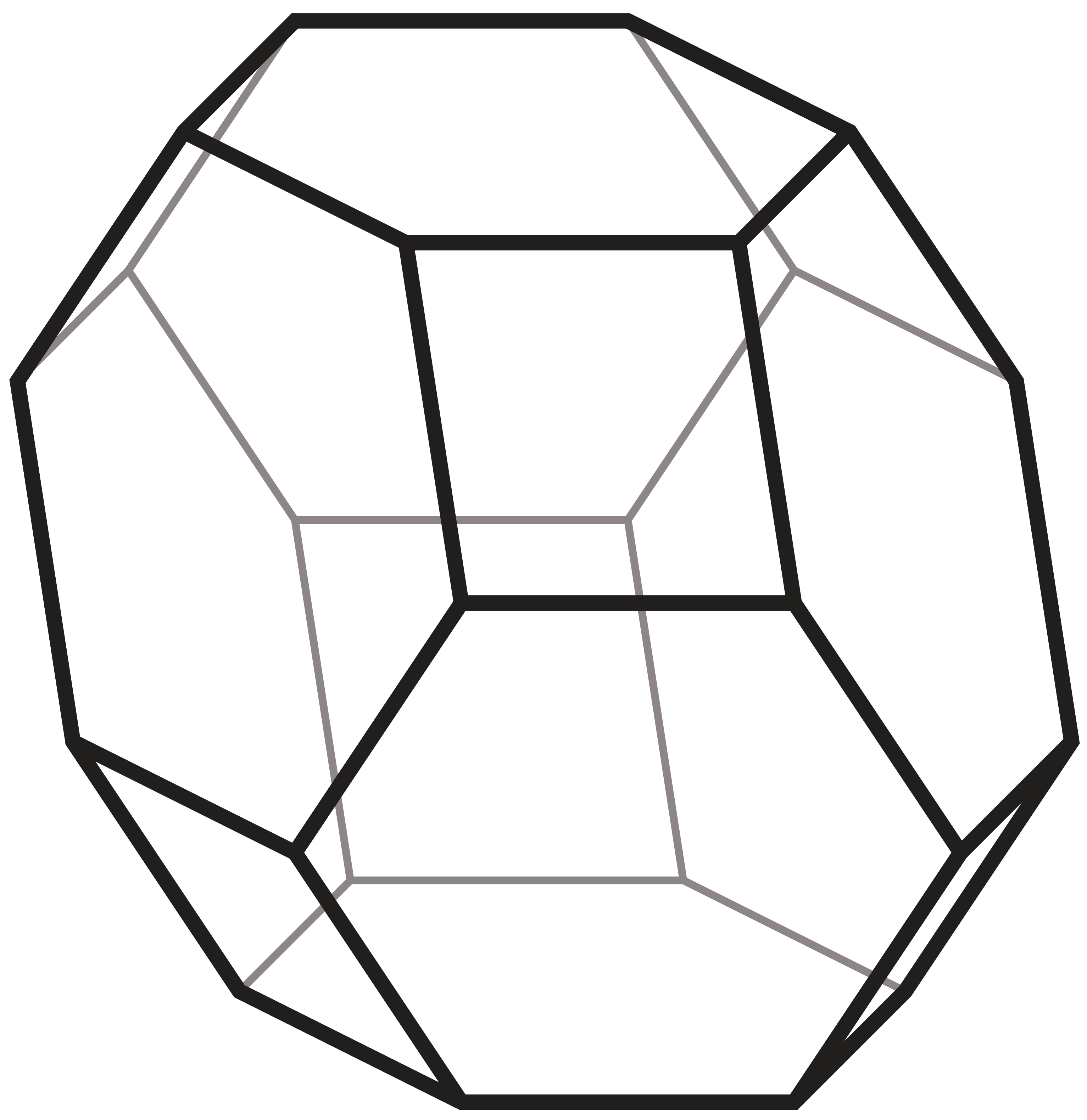
The permutahedron

One way to build it: (Schoute 1914)

Gift wrap these 24 points in 4-D space:

(1,2,3,4) (1,3,2,4) . . . (4,3,2,1)

and you'll get the 4-permutahedron.



In any number of variables/dimension:

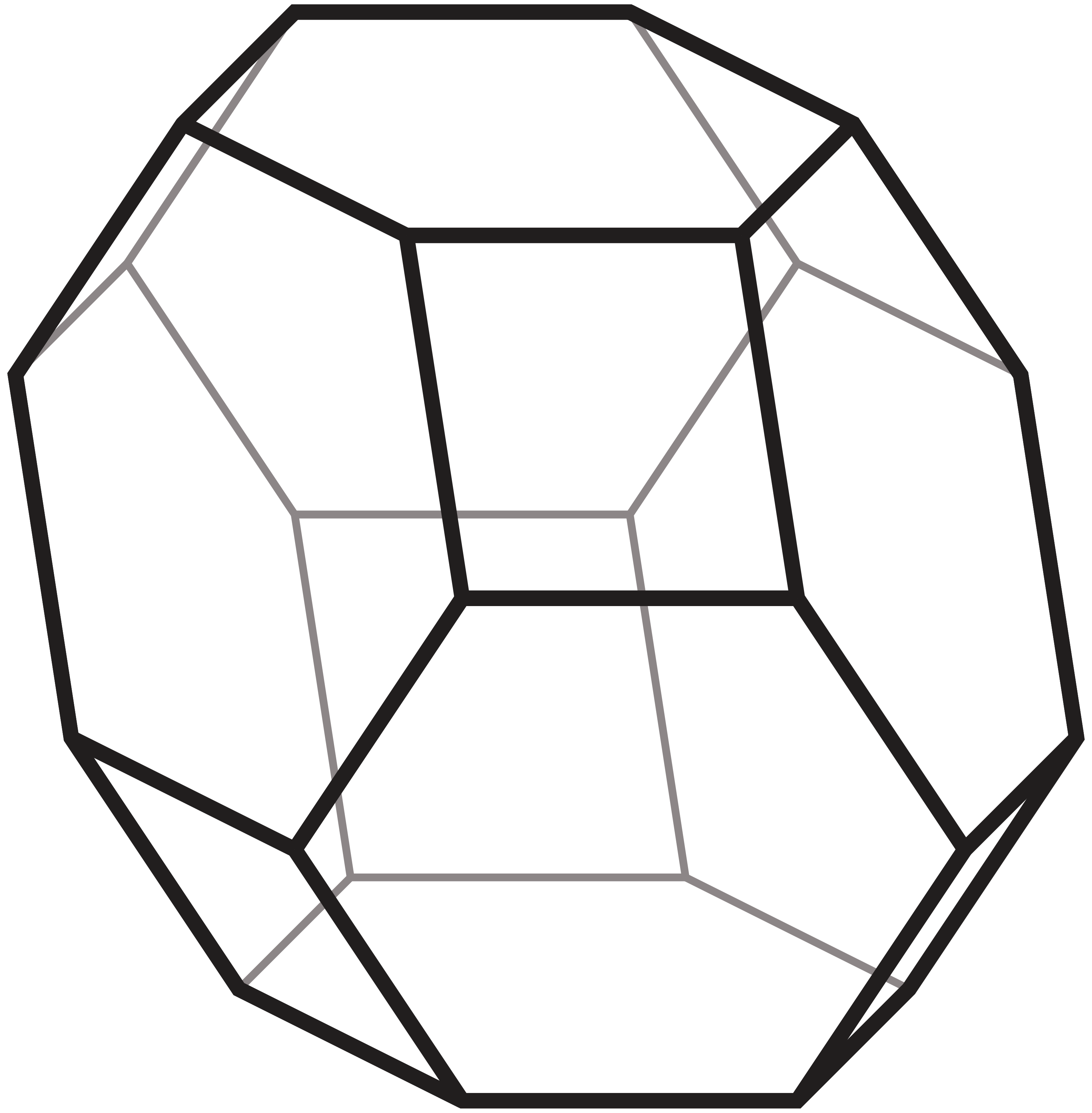
The same construction works!

The order of factors doesn't matter!

The shape of n-commutativity.

Permutahedra in unexpected places

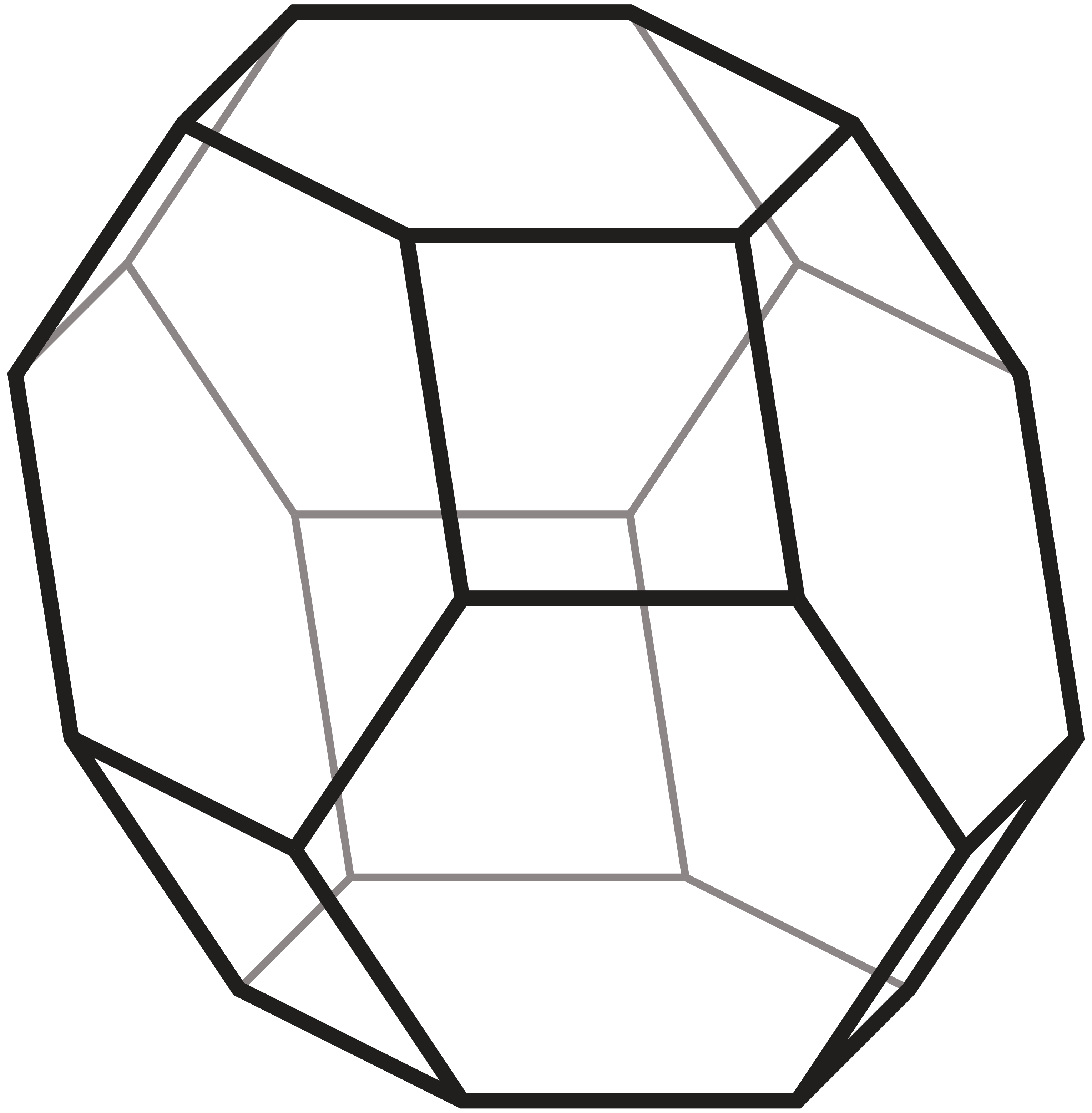
Fluorite crystal.



The shape of 4-commutativity.

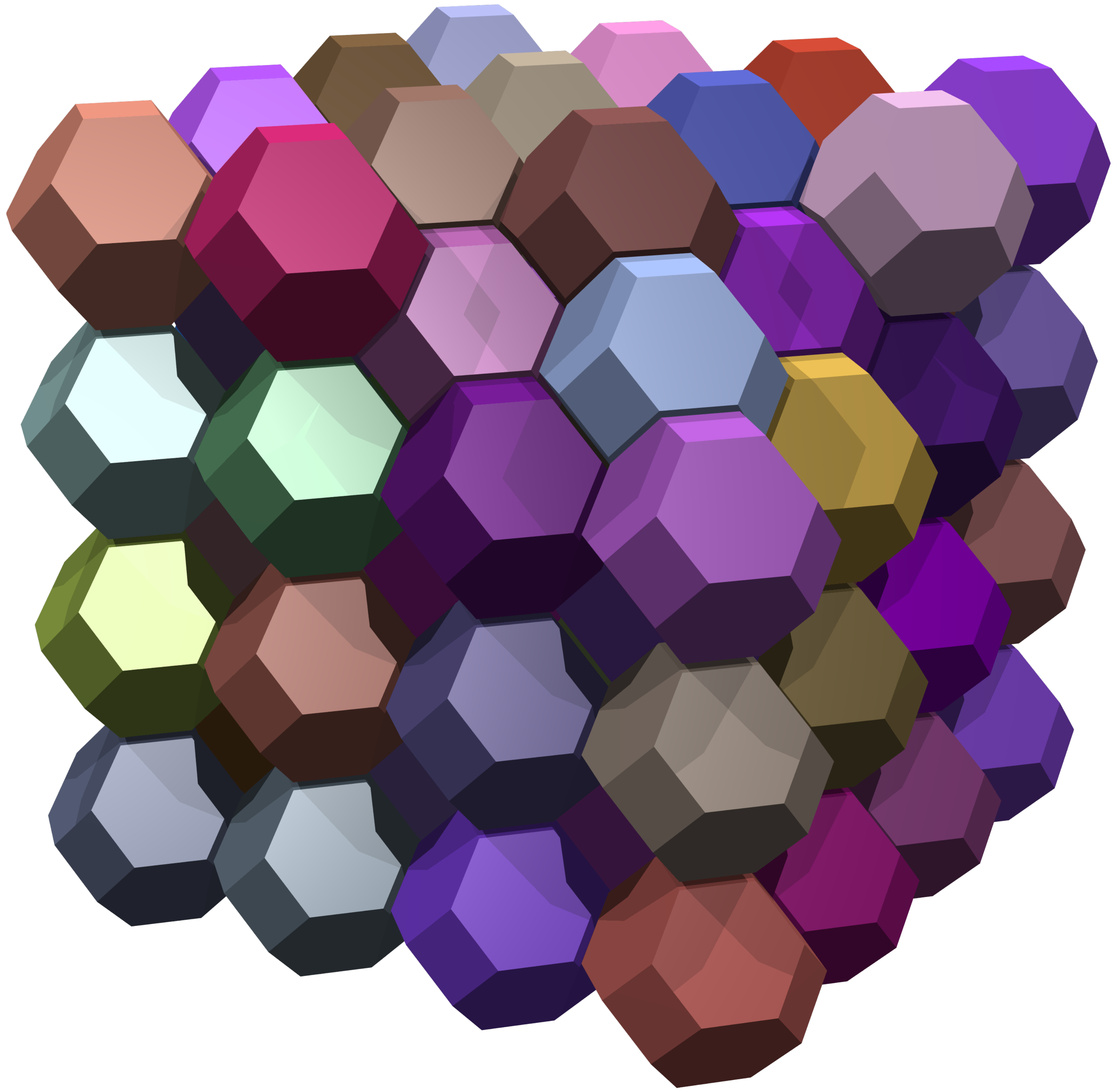
Permutahedra in unexpected places

Goutte-d'Or, Paris

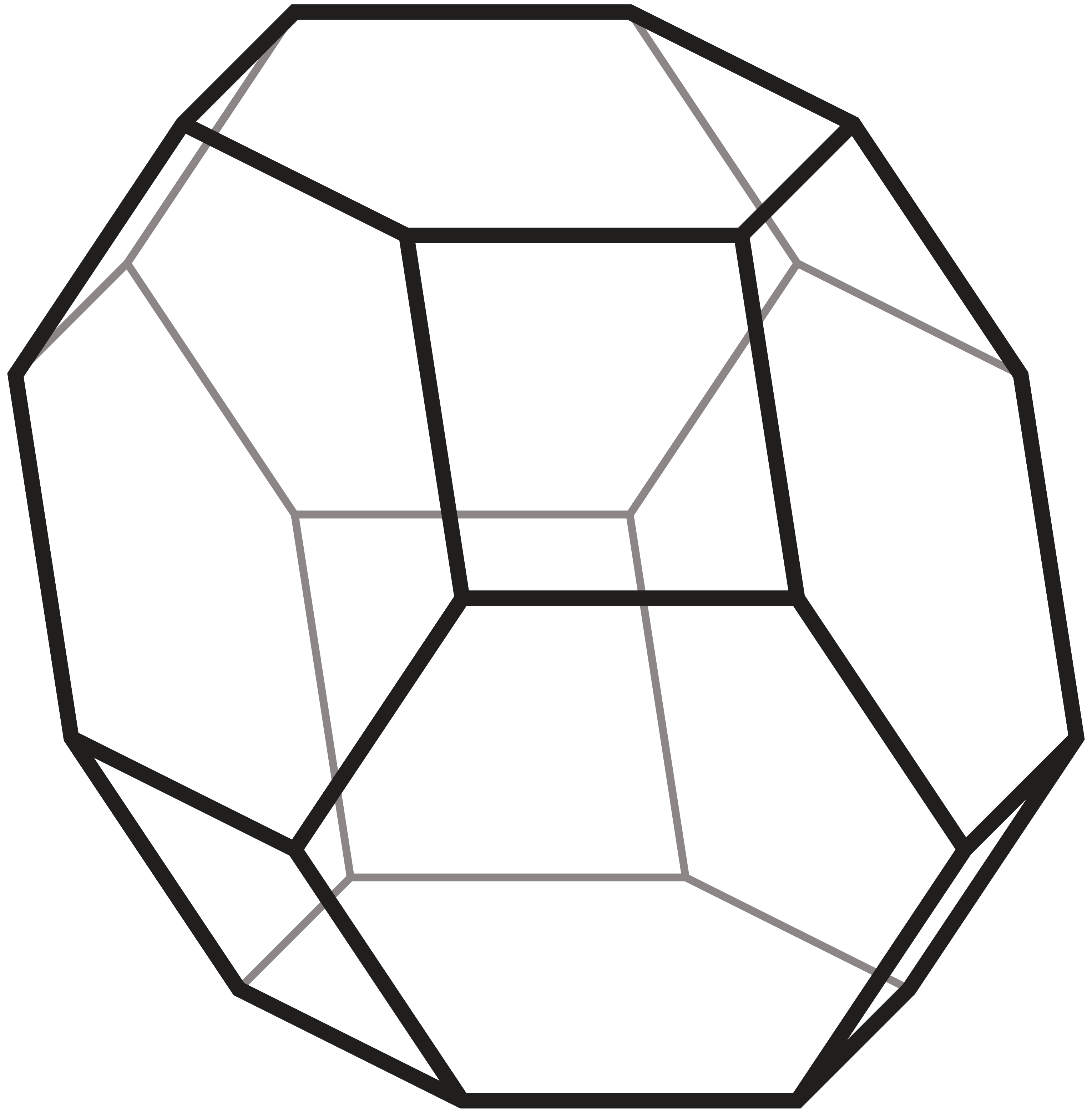


The shape of 4-commutativity.

Permutahedra in unexpected places



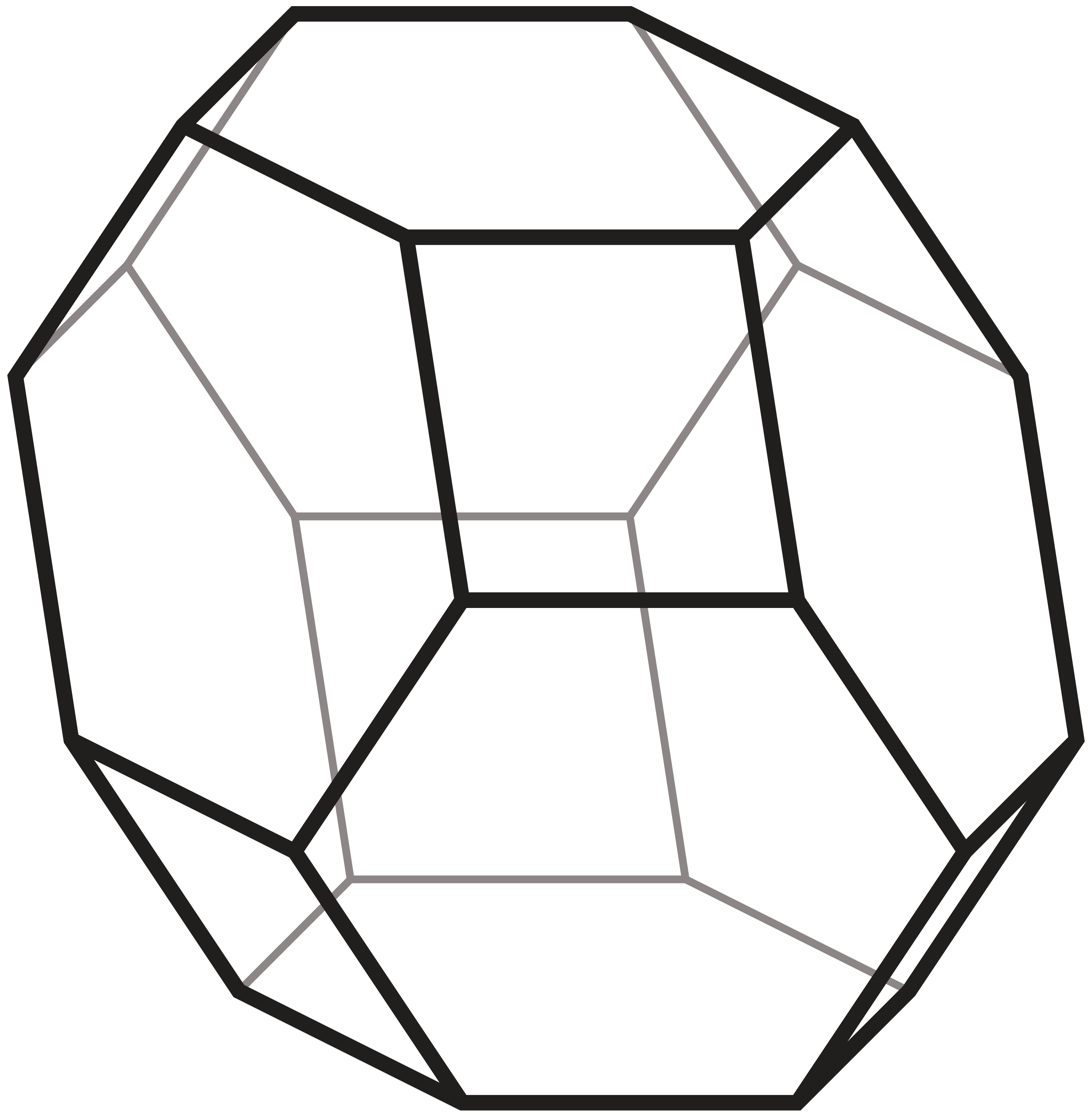
They tile space!



The shape of 4-commutativity.

Permutahedra in unexpected places

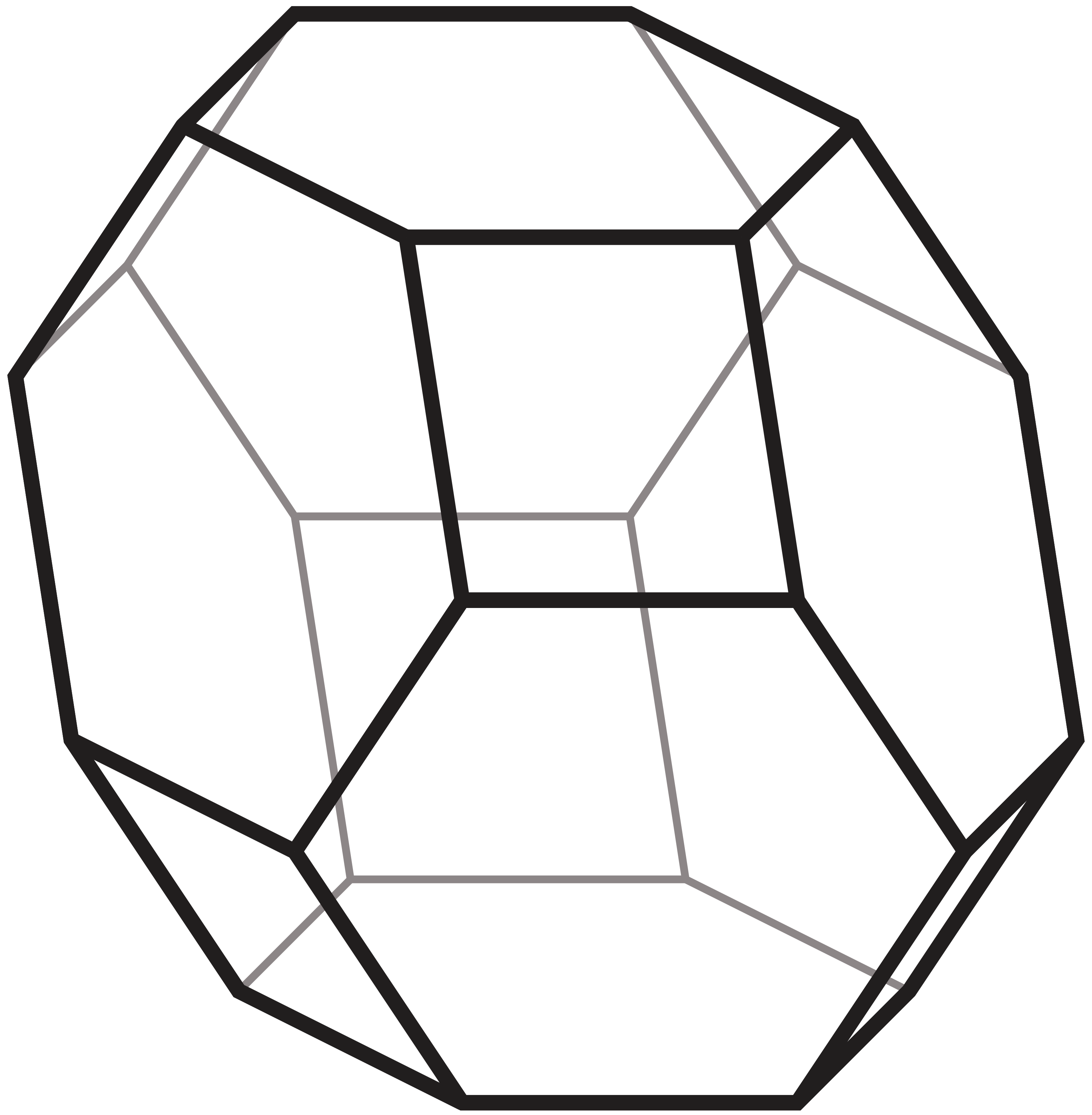
Tarragona, España



The shape of 4-commutativity.

Permutahedra in unexpected places

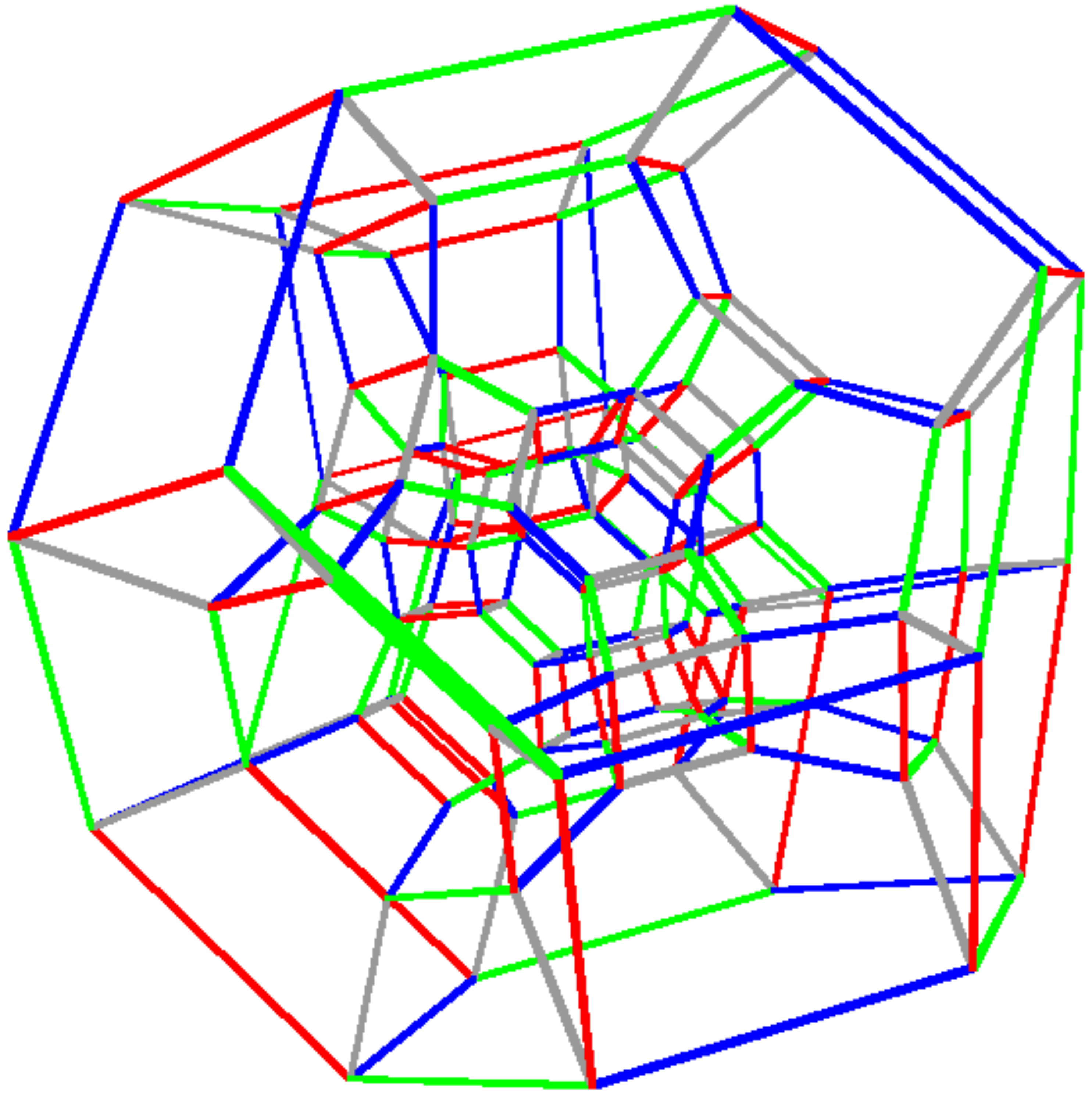
Zeolite sieve.



The shape of 4-commutativity.

Order doesn’t matter: more factors?

The 120 products abcde, ..., edcba are equal:



The shape of 5-commutativity.

The Permutahedron

The n! products of $a_1, a_2, \ldots, a_n$ are equal:

The permutahedron of order n is a beautiful polyhedron. It has:

- dimension: n-1
- vertices: n!
- walls: $2^n - 2$
- volume: n^{n-2}

It tiles (n-1)-dimensional space.

The shape of n-commutativity.

4. GROUPING DOESN'T MATTER

Grouping doesn't matter: $(ab)c=a(bc)$

Idea: No-one can multiply 3 numbers in their head.
Multiply two at a time!

Example: $(a(bc))d$
Non-example: $a(bc)d$

This law says that grouping doesn't matter for 3 numbers.
We want to know why grouping doesn't matter for 4 numbers.

Grouping doesn't matter: $(ab)c = a(bc)$

Let's use the same procedure.

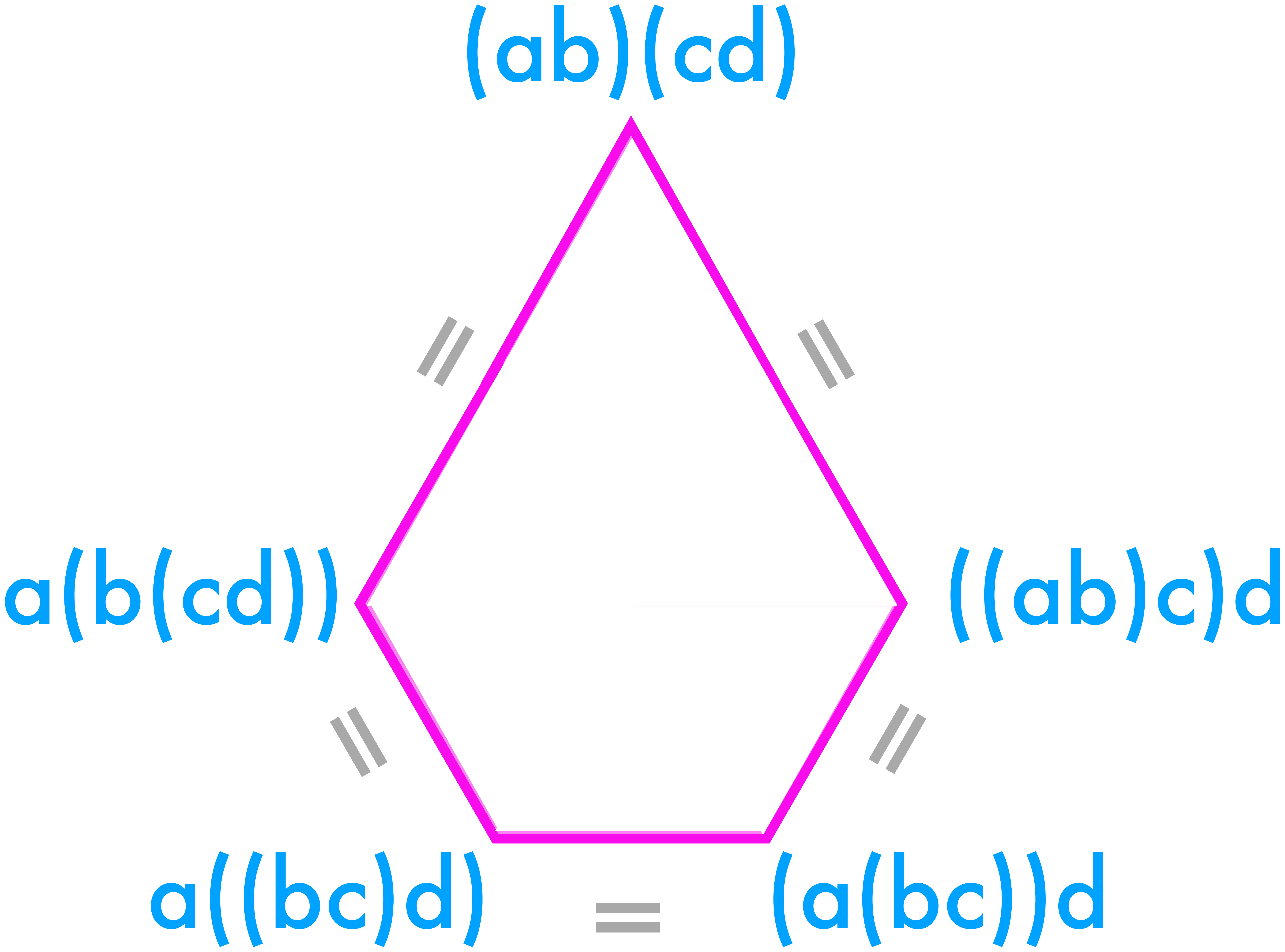
Example: $(a(bc))d$
Non-example: $a(bc)d$

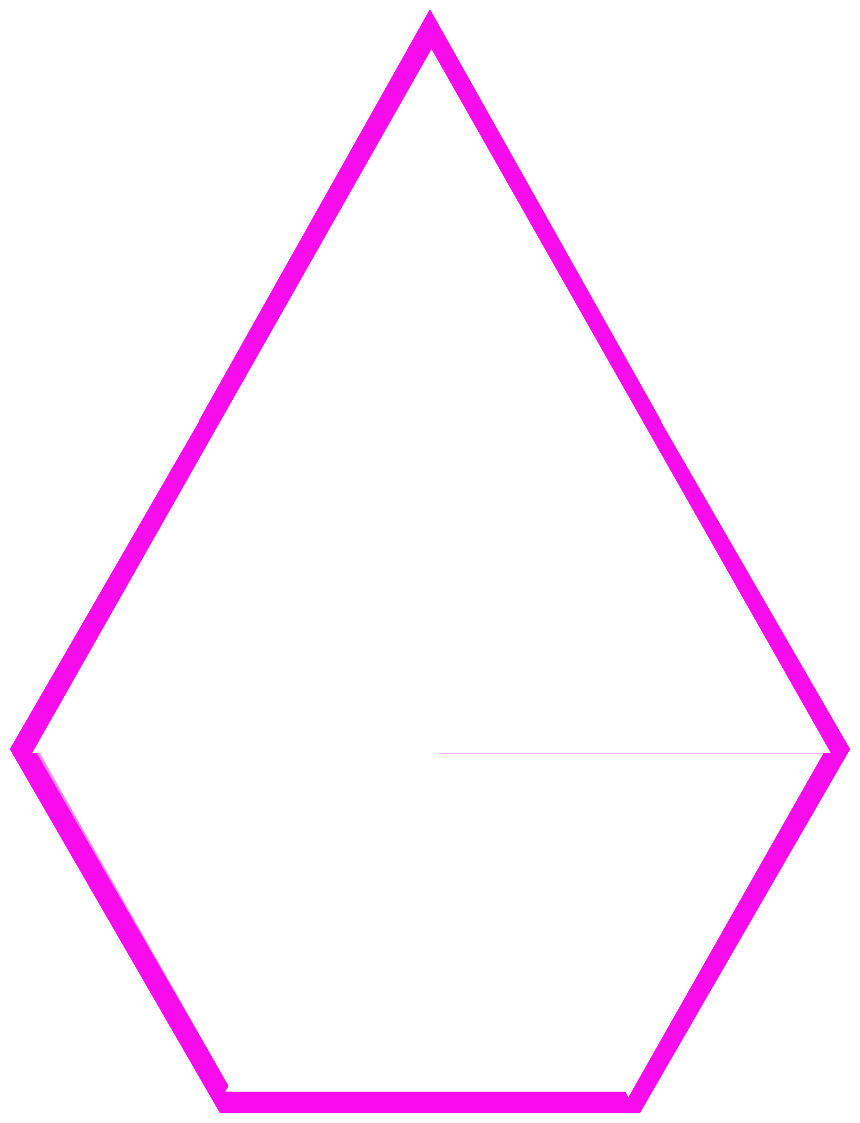
1. What are **all** the ways of grouping the product $abcd$, two factors at a time? (Without changing order of factors.)

2. Why do they **all** give the same answer? Can we prove it?

Grouping doesn't matter:

All 5 groupings of abcd are equal:

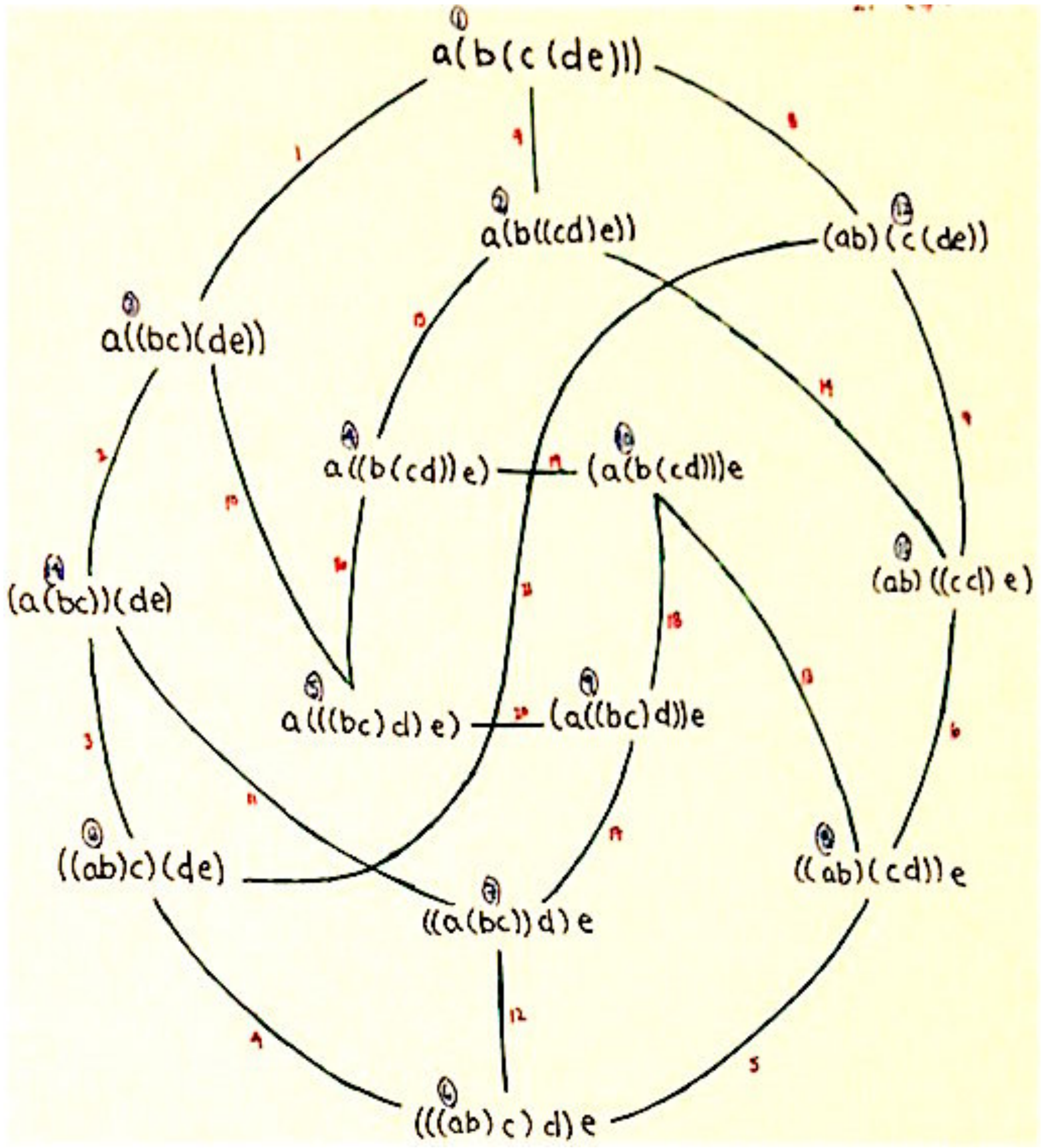


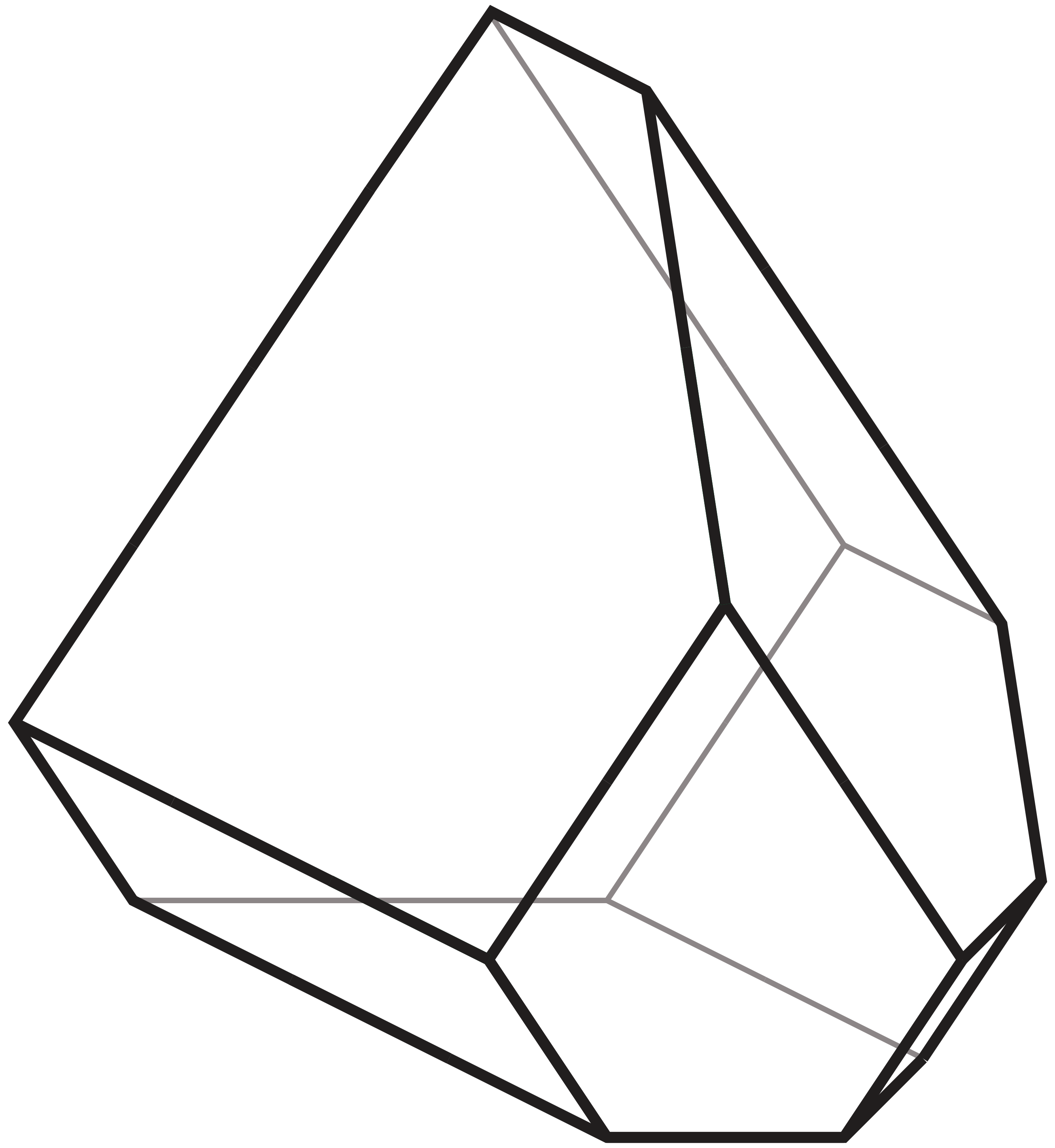


The shape of 4-associativity.

Grouping doesn’t matter:

The 14 groupings of abcde are equal:

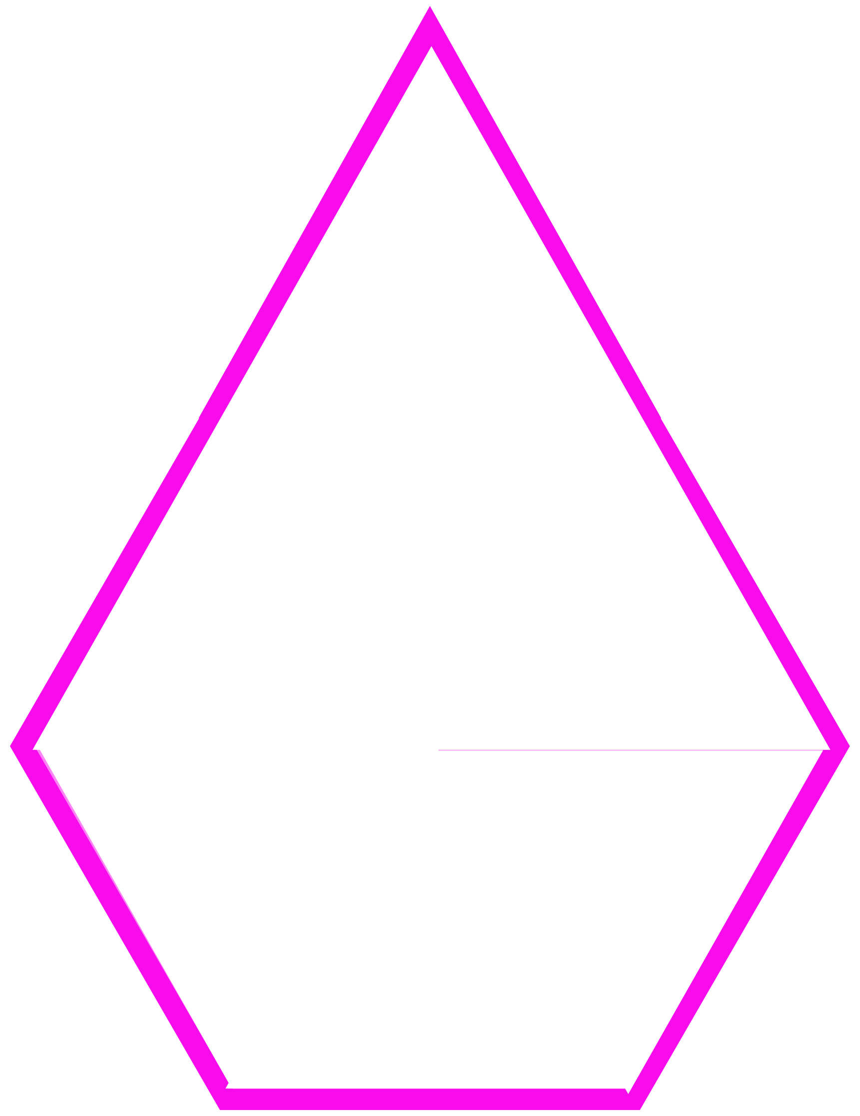




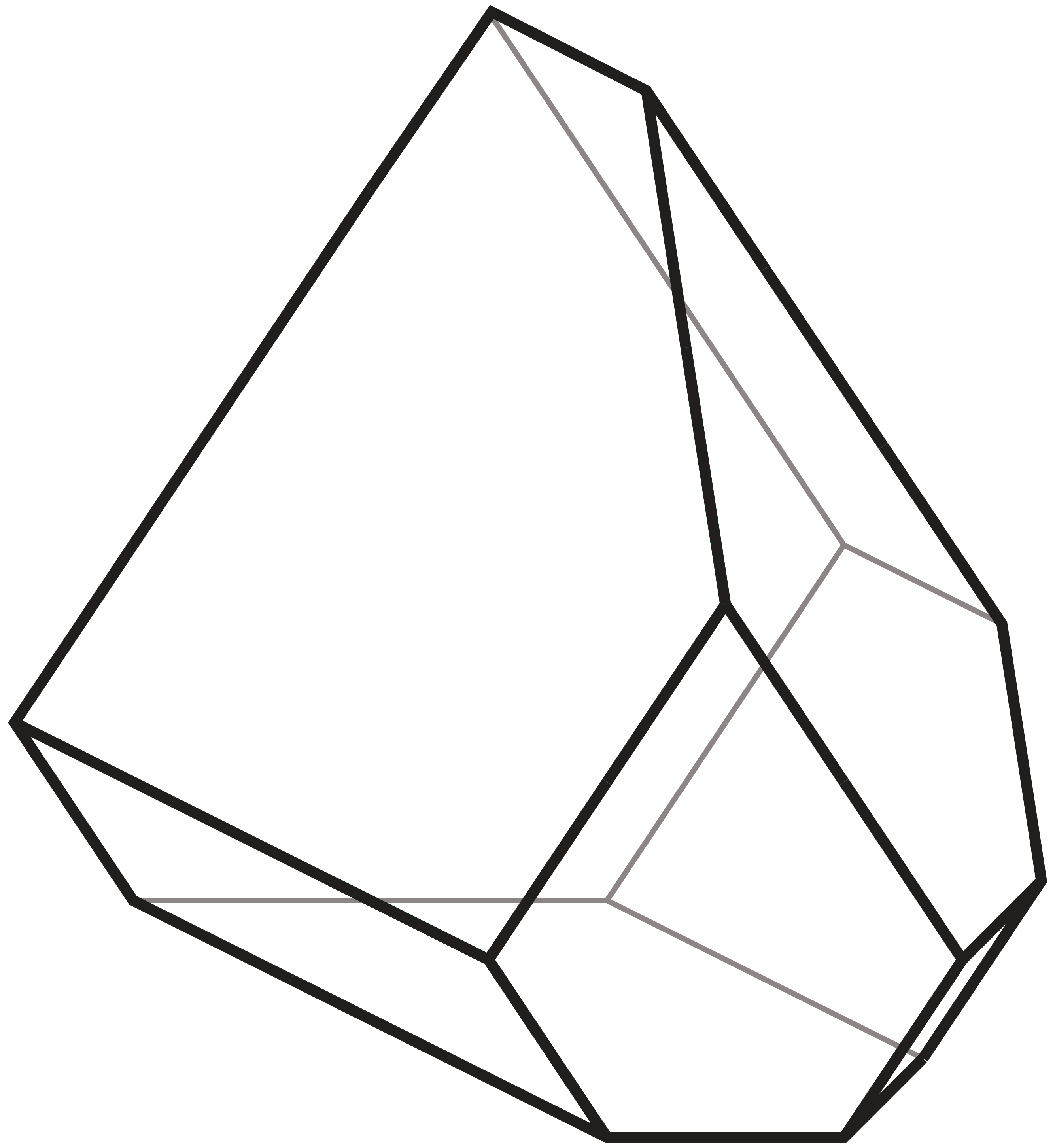
The shape of 5-associativity.

5. THE ASSOCIAHEDRON

The associahedron



The shape of 4-associativity.



The shape of 5-associativity.

The associahedron

The n! groupings of $a_1 a_2 \ldots a_n$ are equal:

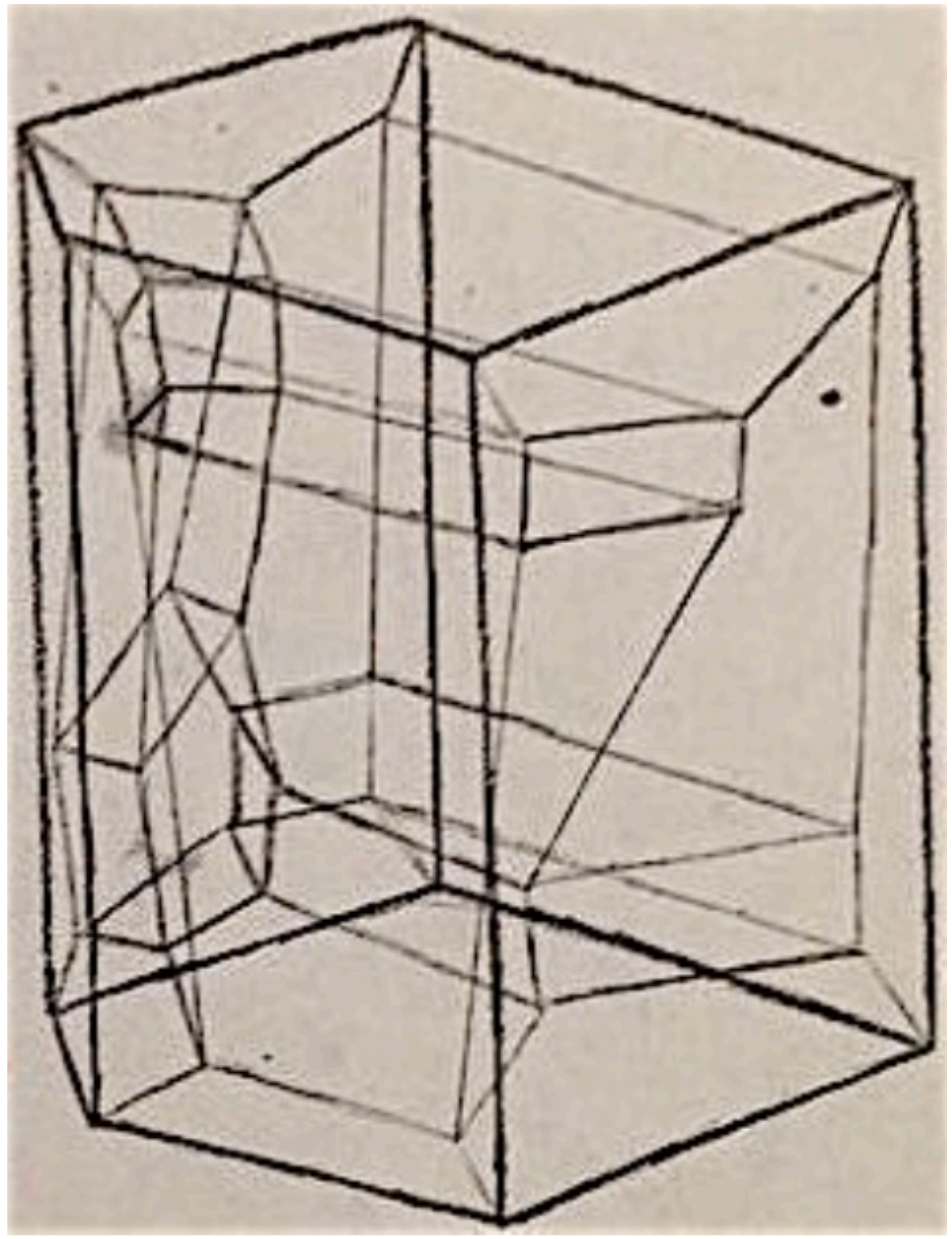
The associahedron of order n is a beautiful polyhedron. It has:

dimension: n-2

vertices: $C_n = (2n)!/n!(n+1)!$

walls: $n(n+1)/2 - 1$

volume: ???

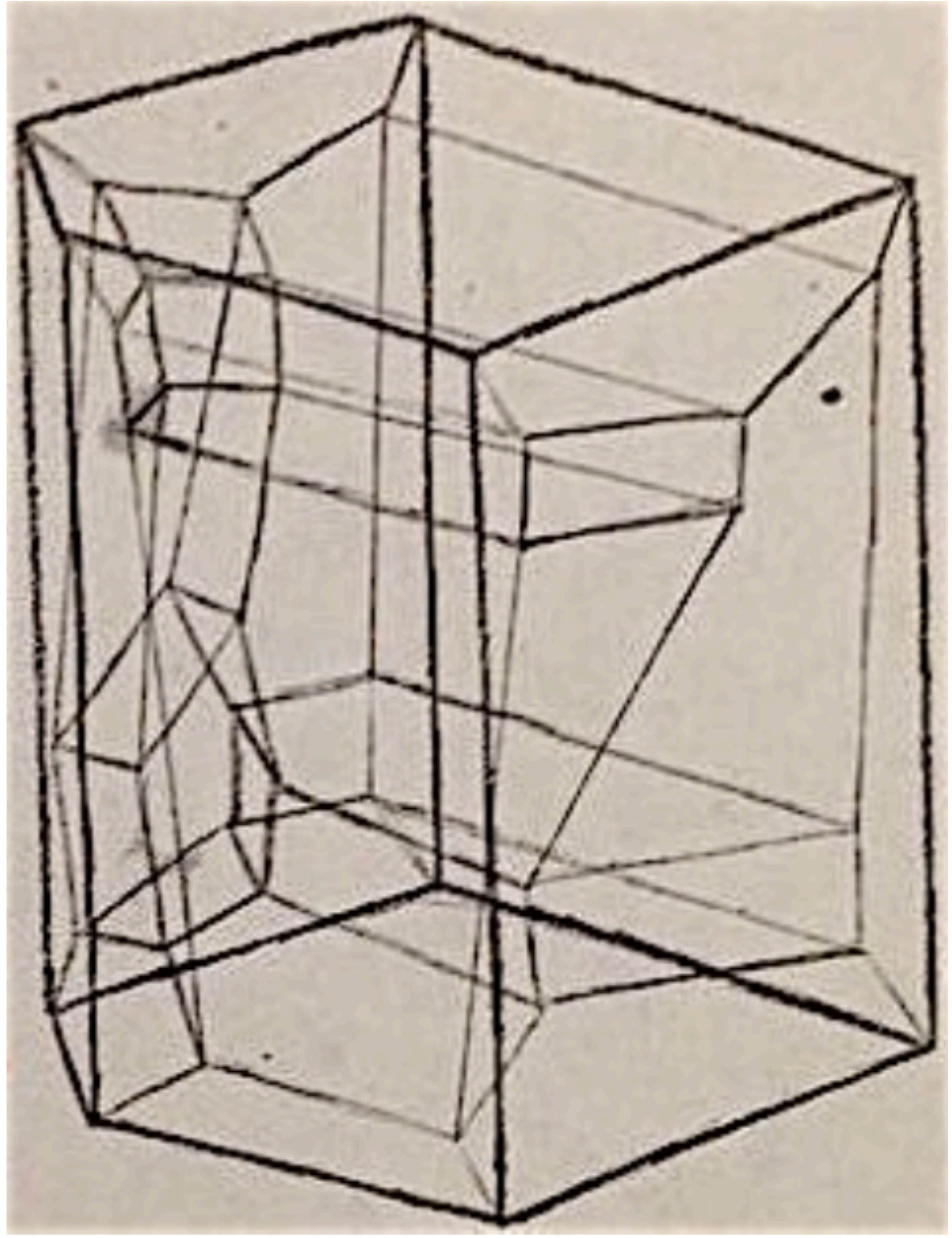


The shape of 6-associativity.

The associahedron: history

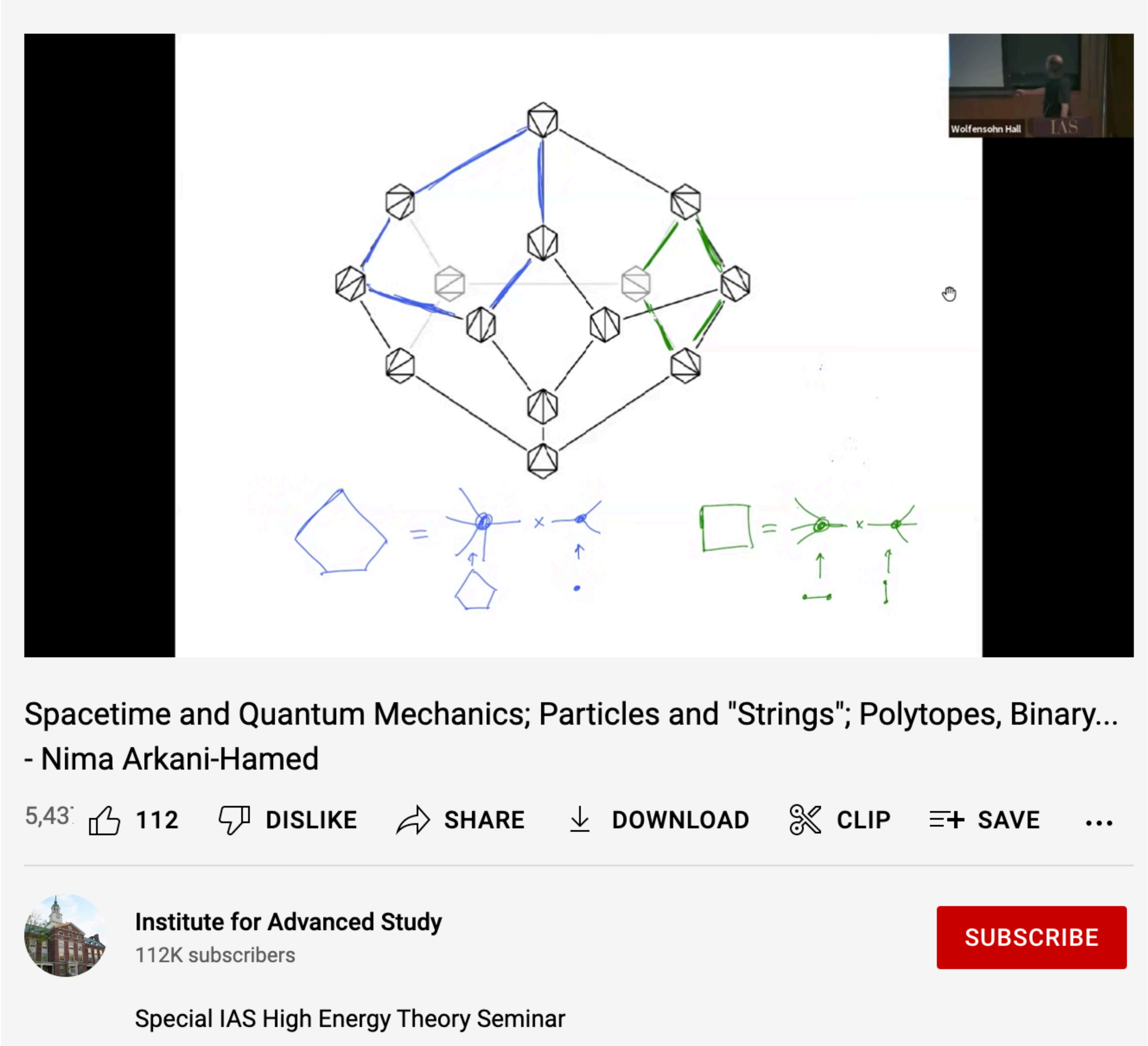
"The associahedron is a mythical polytope representing the parenthesizations of variables."

Stasheff 1963: topology
Haiman 1984: geometry
Loday 2004: algebra
Ceballos et al 2015: combinatorics
Arkani-Hamed et al 2017: physics



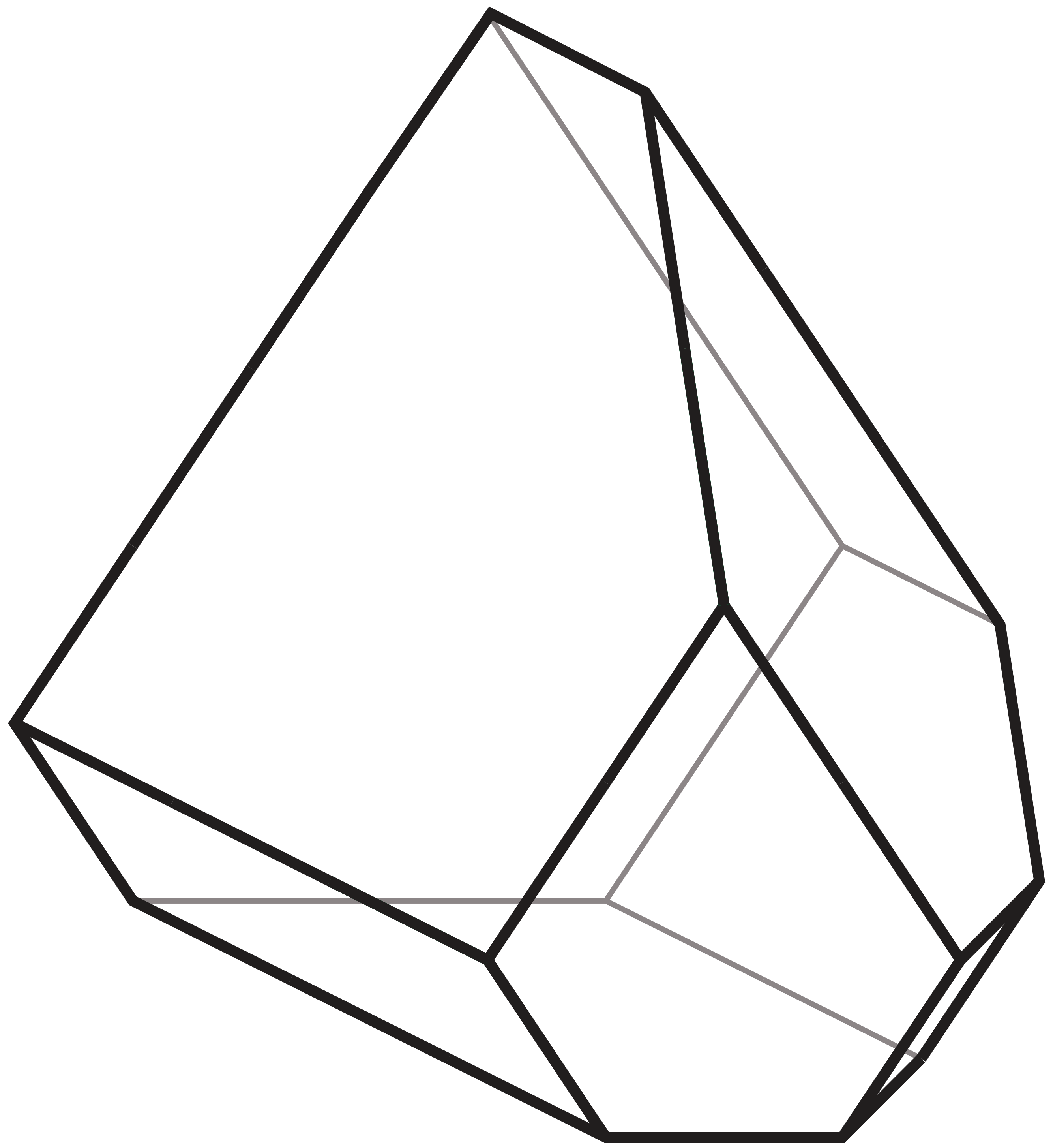
The shape of n-associativity.

Associahedra in unexpected places



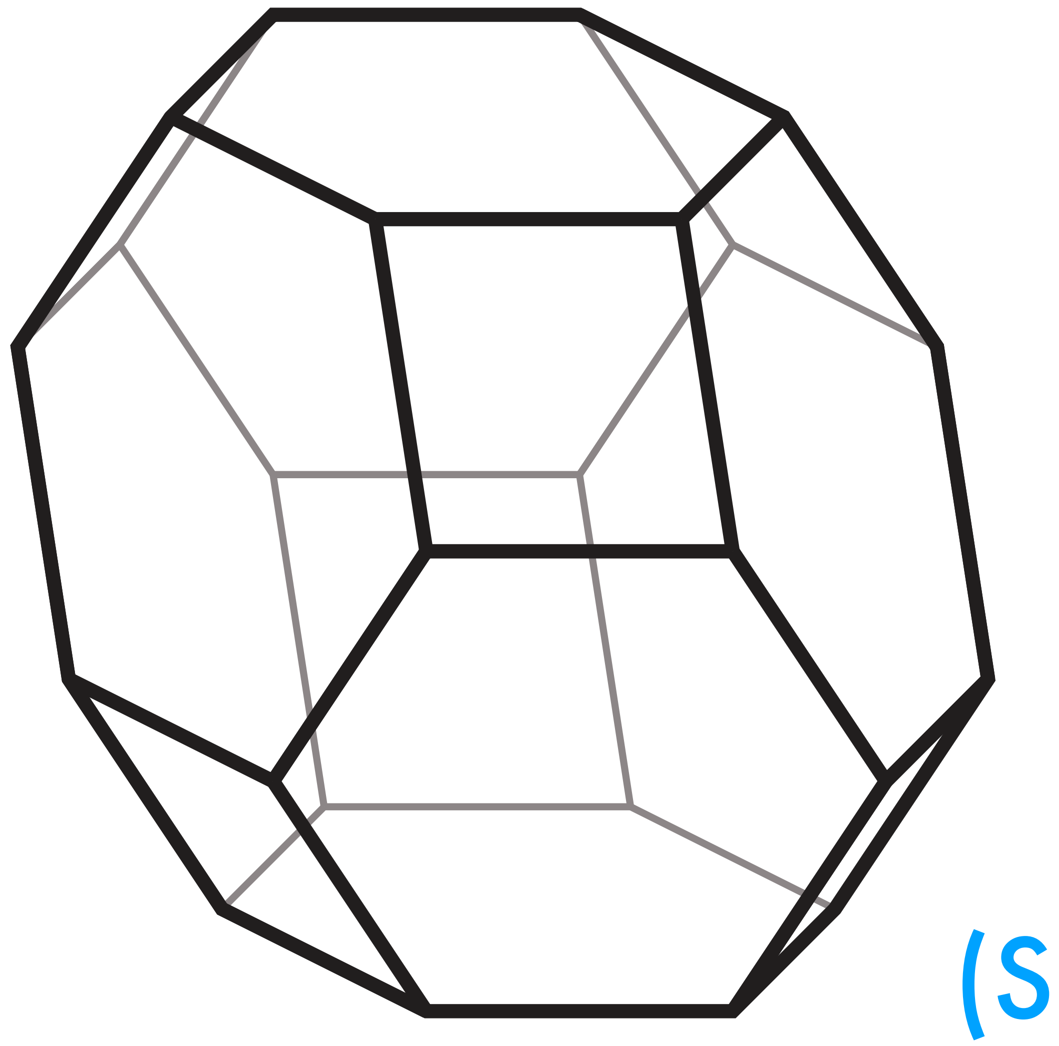
Foundations of spacetime and quantum mechanics

(Arkani-Hamed 2021)



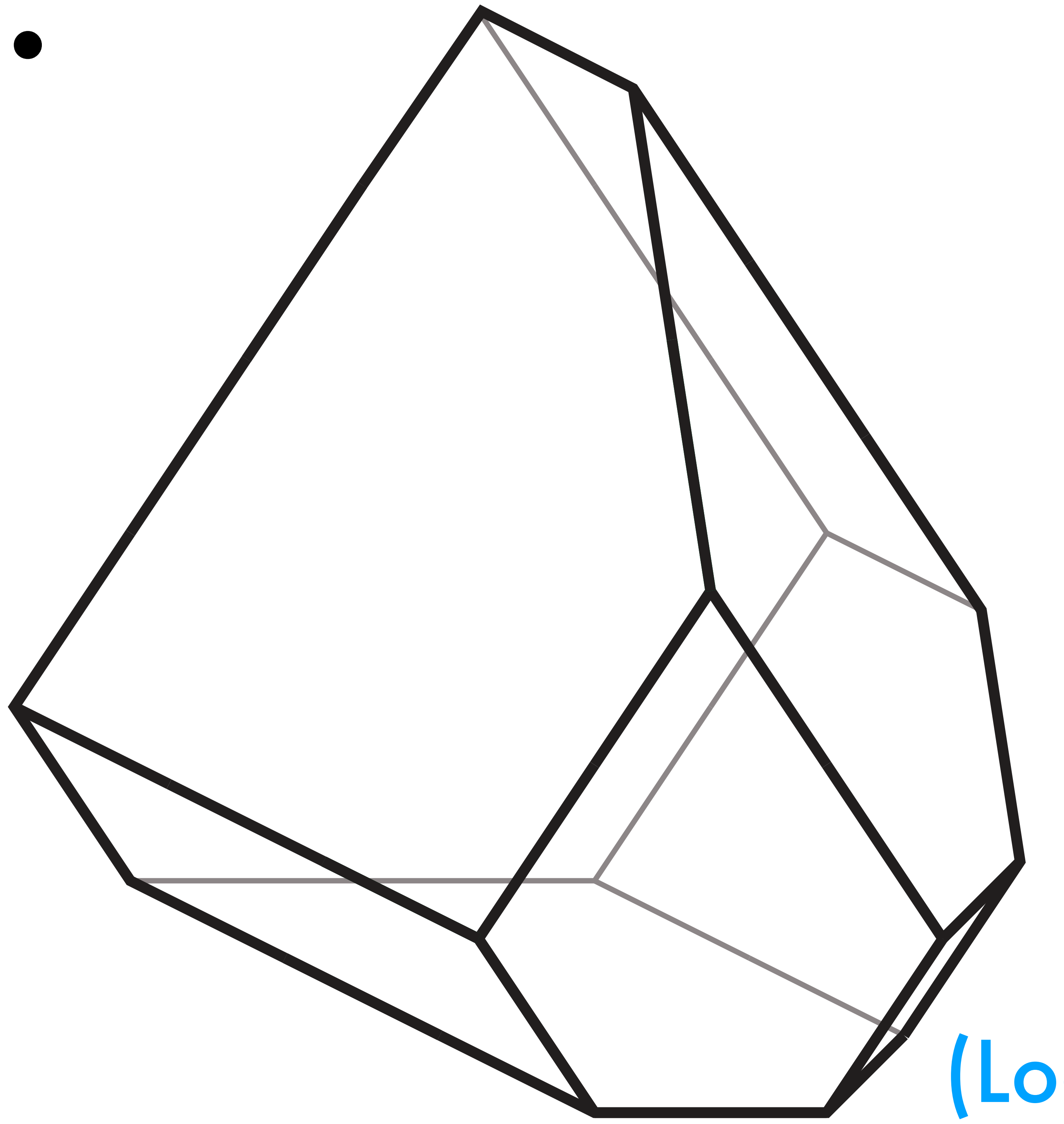
The shape of 5-associativity.

Permutahedron and associahedron, together.



(Schoute 1914)

The shape of 4-commutativity.



(Loday 2004)

The shape of 5-associativity.

You can knock down walls of P(n) to get A(n+1)! (Postnikov 2006)

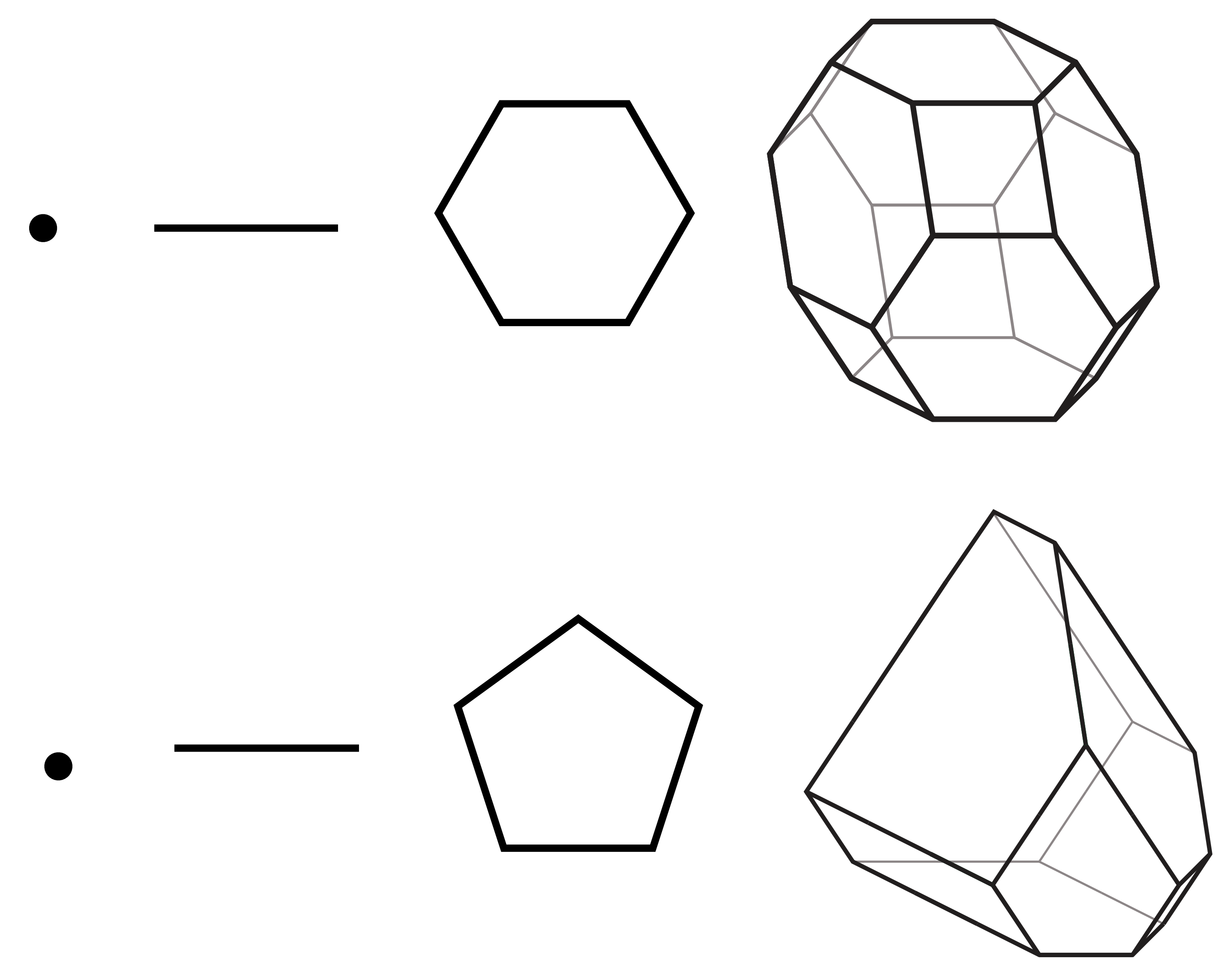
6. RECAP

$$2 \times 3 \times 4 \times 5 = 120$$

Different processes give the same answer!!!

Who is suprised? Who is not surprised?

Why?!?! This is the topic of today.



Mathematics is interconnected, and connected to other areas, in ways we can never predict.

It is always worth thinking very slowly and carefully about what we think we understand!

gracias:

¡¡¡ muchas gracias !!!

questions? comments? reactions?

More information:

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