Hidden patterns: the shape of multiplication

Math Encounters **Museum of Mathematics** September 7, 2022

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Federico Ardila

Federico Ardila is an internationally recognized Colombian-American mathematician who specializes in combinatorics and polytopes. Though he is still young, he has made a name for himself as a dedicated professor and talented therotician. It might come as a surprise that Ardila grew up indifferent to academics, and more interested in becoming a professional soccer player than a professor -- but when he was introduced to mathematics, his life changed.





gracias, team





1. MULTIPLICATION

Question: (For you!) (Please don't scream out your answer yet.)

What answer did you get? How did you do it? How did you feel doing it?

Talk to your neighbor.

 $2 \times 3 \times 4 \times 5 = ?$





Talk to your neighbor.

$2 \times 3 \times 4 \times 5 = 120$

Different processes give the same answer!!! Who is suprised? Who is not surprised?

Why do we get the same answer?!?! Today's topic.



Two simple laws with complicated names: (or: why we get the same answer)

Commutative Law: $a \times b = b \times a$

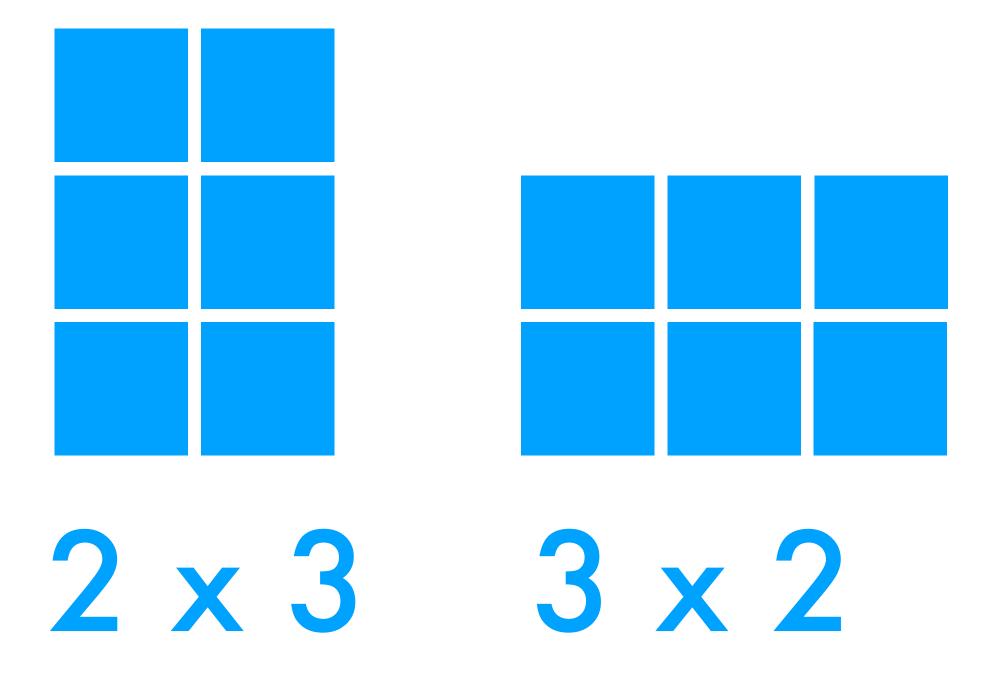
Associative Law: $(a \times b) \times c = a \times (b \times c)$

"The order of operations doesn't affect the result."



2. ORDER DOESN'T MATTER

Order doesn't matter: $a \times b = b \times a$ Why?!?! One illustration:



Is this a proof?

Not to a mathematician. $ls 202 \times 117 = 117 \times 202?$ We need to think more slowly.



Order doesn't matter: $a \times b = b \times a$

Why? Terry Tao's Analysis 1:

Lemma 2.3.2 (Multiplication is commutative). Let n, m be natural numbers. Then $n \times m = m \times n$.

Proof. See Exercise 2.3.1.

We learn a lot from thinking slowly about what we think we understand. - We build solid foundations. - We discover cool things we had missed.



We now discuss how addition interacts with positivity

Definition 2.2.7 (Positive natural numbers). A natural number n is said to be positive iff it is not equal to 0. ("iff" is shorthand for "if and only if" - see Section A.1). Proposition 2.2.8. If a is positive and b is a natural number, then a+bis positive (and hence b+a is also, by Proposition 2.2.4).

Proof. We use induction on b. If b = 0, then a + b = a + 0 = a, which is positive, so this proves the base case. Now suppose inductively that a + b is positive. Then a + (b + i) = (a + b) + i, which cannot be zero by Axiom 2.3, and is hence positive. This closes the induction.

Corollary 2.2.9. If a and b are natural numbers such that a + b = 0, then a = 0 and b = 0.

rer. witting 2.2.11 (Ordering of the natural numbers). Let n and m be al numbers. We say that n is greater than or equal to m, and write n or $m \leq n$, iff we have n = m + a for some natural number a. by that n is strictly greater than m, and write n > m or m < n, iff

Proof. See Exercise 2.2.2.



20 1	2. St	arting at th	e beginning	: the natural i	umbe
should technically be a is a template for prod- being a single axiom in is far beyond the scop logic.)	lucing n its e	; an (infinite own right. "	e) number (Fo discuss t	of axioms, rath his distinction	er tha furthe
The informal initial P(n) is such that $P(0$ then $P(n++)$ is true. P(n) is such that $P(0)$, then P(n) is true. $P(1)we see that P(0), P(1)innecessary "elementthis fact. Apply Axis:"innecessary" elementthen P(n++) is true.And P(1) integer", i.e., an integerinteger", i.e., an integerinteger", i.e., an integerinteger", and P(3) byparticular, 0.5 cannolthen P(n++) is true.The principle of inductionthe "true" natural num-true" natural num-tion P(n) is true for everywe will see many procewe will see many proce$	0) is Ther (1++) 1), F never shoul ts suc om 2. er plu Thus e, no : be a have r is bu is su mbers aduct r natu	true, and s a since $P(0$ is P(2) is $P(2)$ is P(2), $P(3)$, $P(3)$, $P(4)$, P(3), $P(3)$, $P(3)$, $P(4)$, P(5), $P(3)$, $P(4)$, $P(3)$, P(3), $P(3)$, $P(3)$ $P(3)$, P(3), $P(3)$, P(3), $P(3)$, P(3), $P(3)$, P(3), P(3), $P(3)$, P(3), P(3	uch that w) is true, $Ptrue. Repe-etc. are achude that ifor numberindeed, oneoperty P(rP(0)$ is tru- δ asserts th umber can umber. This such notice prohibit any aring in N s a way to n. Thus i	hencver $P(n)$ (0++) = P(1) (0++) = P(1) (0++) = P(1) (0++) = P(1), for inst (0++) = n "is not (0++) = n "is not (is true is true finitely ver the ance, contain roof" of a hall is true e for a ger. I of quit , "hall how the er that roopert
Proposition 2.1.11. number n.	A ce	rtain prope	rty $P(n)$ is	true for every	natun
Proof. We use inducti prove $P(0)$. (Insert pro- is a natural number, a P(n++). (Insert proo- This closes the inducti	oof of and <i>P</i> of of .	P(0) here) P(n) has alr P(n++), as	. Now supp eady been p suming the	ose inductively proven. We no it $P(n)$ is true	y that w prov , here
Of course we will or order in the above generally be somethin variants of induction v	type ig like	of proof, by the above	it the proo form. The	s using induct e are also son	ion wi ie othe



(a) (Order is reflexive) a ≥ a.

(a) (Order is reprinted a ≥ a.
(b) (Order is transitive) If a ≥ b and b ≥ c, then a ≥ c.
(c) (Order is anti-symmetric) If a ≥ b and b ≥ a, then a = b. (d) (Addition preserves order) $a \ge b$ if and only if $a + c \ge b + c$

 $(e) \ a < b \ if \ and \ only \ if \ a++ \leq b.$ $(f) \ a < b \ if \ and \ only \ if \ b = a + d \ for \ some \ positive \ number \ d.$ Proof. See Exercise 2.2.3. Proposition 2.2.13 (Trichotomy of order for natural numbers). and b be natural numbers. Then exactly one of the following state is true: a < b, a = b, or a > b. Proof. This is only a sketch of the proof; the gaps will be filled in Exe rise 2.2.4

rows ..., as a only a short of the proof the gaps will be filled in Eerr. There were then we cannel have now that one of the starsments a < b a < b a < b has b in the starsment have the starsment is a < b a < b a < b > b hading at the same time. If a < b then a ≠ b by by finding, and f a > b then a ≠ b by definition. If a > b and a < b then the starsment is the starsme

The properties of order allow one to obtain a stronger version of the principle of induction:

P(m) is also true. (In particular, this means that $P(m_0)$ is true, since in this case the hypothesis is vacuous.) Then we can conclude that P(m)is true for all natural numbers $m \ge m_0$. Remark 2.2.15. In applications we usually use this principle with $m_0 = 0$ or $m_0 = 1$ Proof. See Exercise 2.2.5.

-Exercises --Exercises 2.21. Prove Proposition 2.25. (Hint: fix two of the variables and induct on the birth) Exercise 2.22. Prove Fourname 2.210. (Hint: use induction.) Exercise 2.22. Prove Propulstates 2.212. (Hint: you will need many of the Exercise 2.22. Justify ther three statements marked (why?) in the proof of Propulstion 2.2.13. Exercises 2.22. Then the three statements marked (why?) in the proof of Propulstion 2.2.13. Proposition 2.2.10. Exercise 2.2.5. Prove Proposition 2.2.14. (Hint: define Q(n) to be the property that P(m) is true for all $m_0 \leq m < n$; note that Q(n) is vacuously true when Exercise 2.2.6. Let n be a natural number, and let P(m) be a property per $\sum_{i=1}^{n} \frac{1}{m} \sum_{i=1}^{n} \frac{1}{m} \sum_{i=$ is true. Suppose that P(n) is also true. Prove that P(m) is true for all natura numbers $m \le n$; this is known as the principle of backwards induction. (Hint apply induction to the variable n.)

In the previous section we have proven all the basic facts that we know to be true about addition and order. To aver space and to avoid behaloning concerning addition and order that we are familiar with whome further connerm. Thus for instance we may write things like a + b + c = c + b + awithout further provide the state of the state of

Thus for instance $0 \times m = 0$, $1 \times m = 0 + m$, $2 \times m = 0 + m + n$ etc. By induction one can easily verify that the product of two nature numbers is a natural number. Lemma 2.3.2 (Multiplication is commutative). Let n, m be nature numbers. Then $n \times m = m \times n$.

Proof. See Exercise 2.3.1. We will now abbreviate $n \times m$ as nm, and use the usual convention that multiplication takes precedence over addition, thus for instance ab + c means $(a \times b) + c$, not $a \times (b + c)$. (We will also use the usus notational convertions of precedence for the other arithmetic operation when they are defined later, to save on using parentheses all the time: Lemma 2.3.3 (Positive natural numbers have no zero divisors). n,m be natural numbers. Then $n \times m = 0$ if and only if at least one n,m is equal to zero. In particular, if n and m are both positive, th nm is also positive. Proposition 2.3.4 (Distributive law). For any natural num-we have a(b + c) = ab + ac and (b + c)a = ba + ca. a construction of the structure of the

Order doesn't matter: $a \times b = b \times a$ Why is 2x3x4x5 = 5x2x4x3?

Why is abcd = dacb?

The commutative law is about 2 factors, not 4. Go slow! abcd = abdc = adbc = dabc = dacb

Ok cool! But what about all the other orders of a,b,c,d? Why do they all give the same answer?

Questions: (For you!) 1. What are all the possible orders for multiplying a,b,c? 2. Why do they all give the same answer? Can you prove it?

Talk to your neighbor.

Order doesn't matter: ab = ba

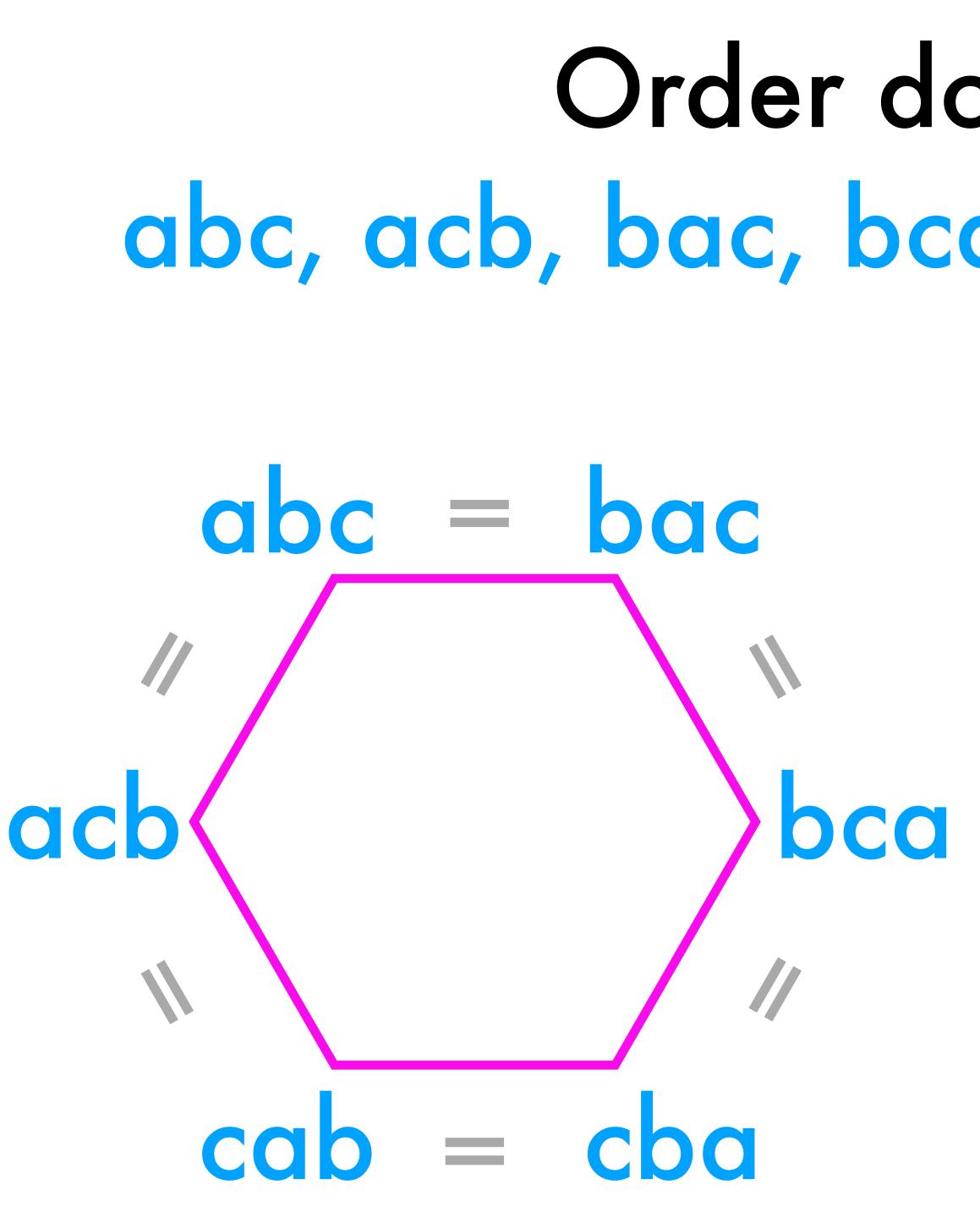




Talk to your neighbor.

Youth Speaks Life is Living





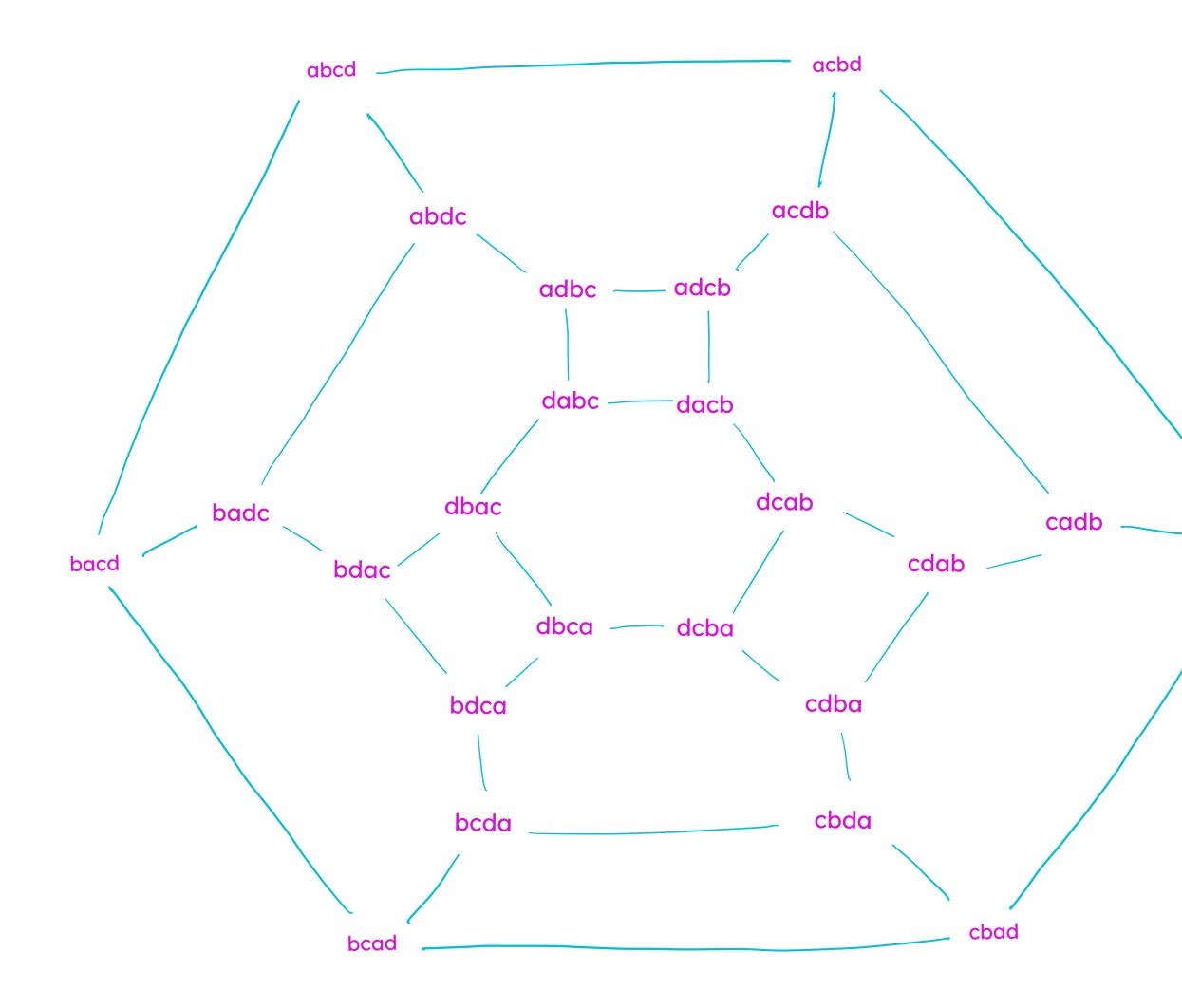
Order doesn't matter: abc, acb, bac, bca, cab, cba are equal:



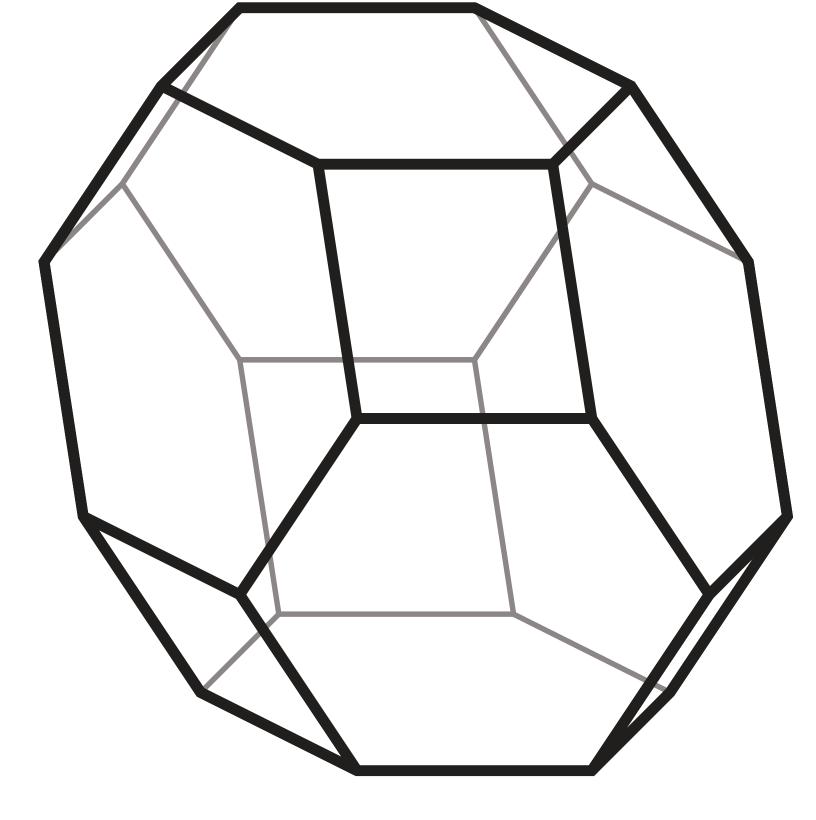
Order doesn't matter: All 24 products abcd, ..., dcba are equal.

Let's apply the same method!

Order doesn't matter: All 24 products abcd, ..., dcba are equal:



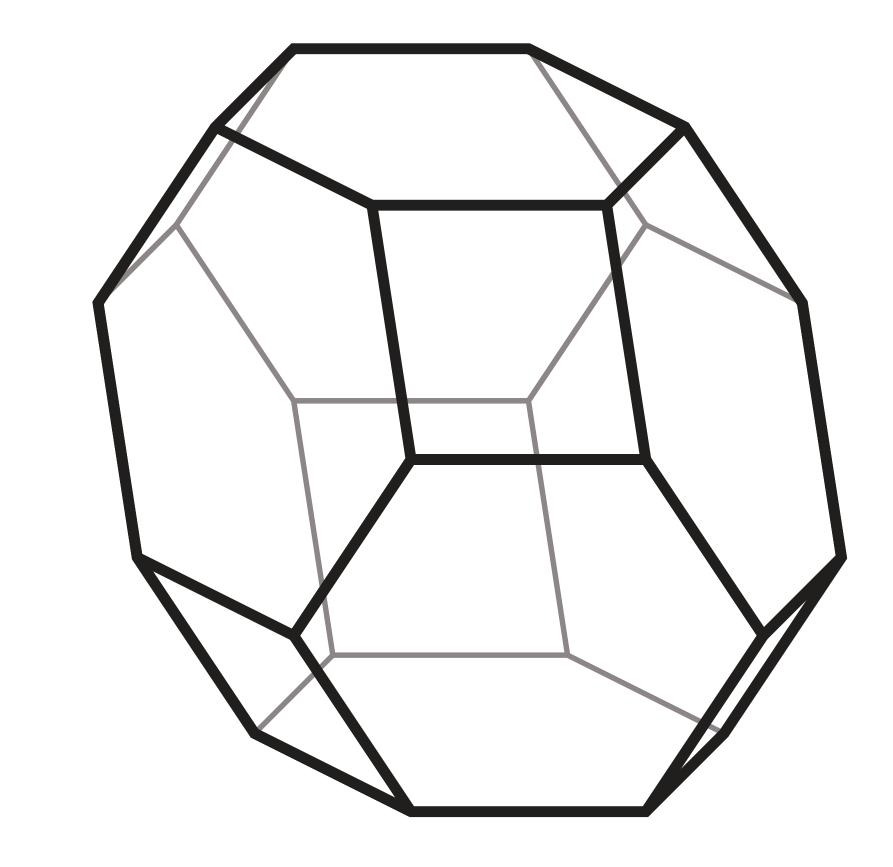




3. THE PERMUTAHEDRON

The permutahedron



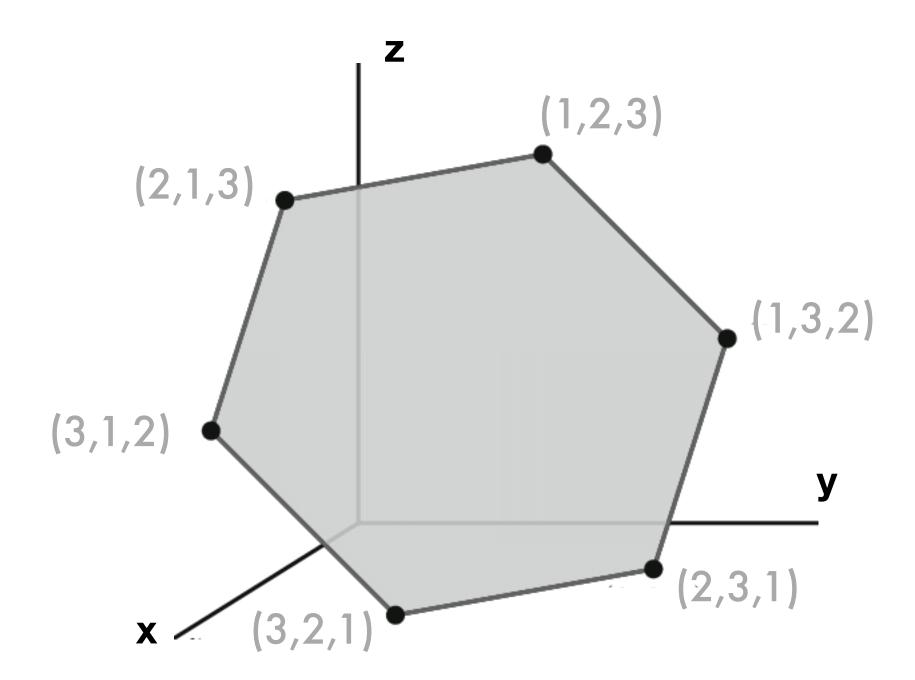




The permutahedron

The shape of 3-commutativity.

One way to build it: (Schoute 1914) Gift wrap these 6 points in 3-D space: (1,2,3) (1,3,2) (2,1,3) (2,3,1) (3,1,2) (3,2,1) and you'll get the 2-permutahedron.

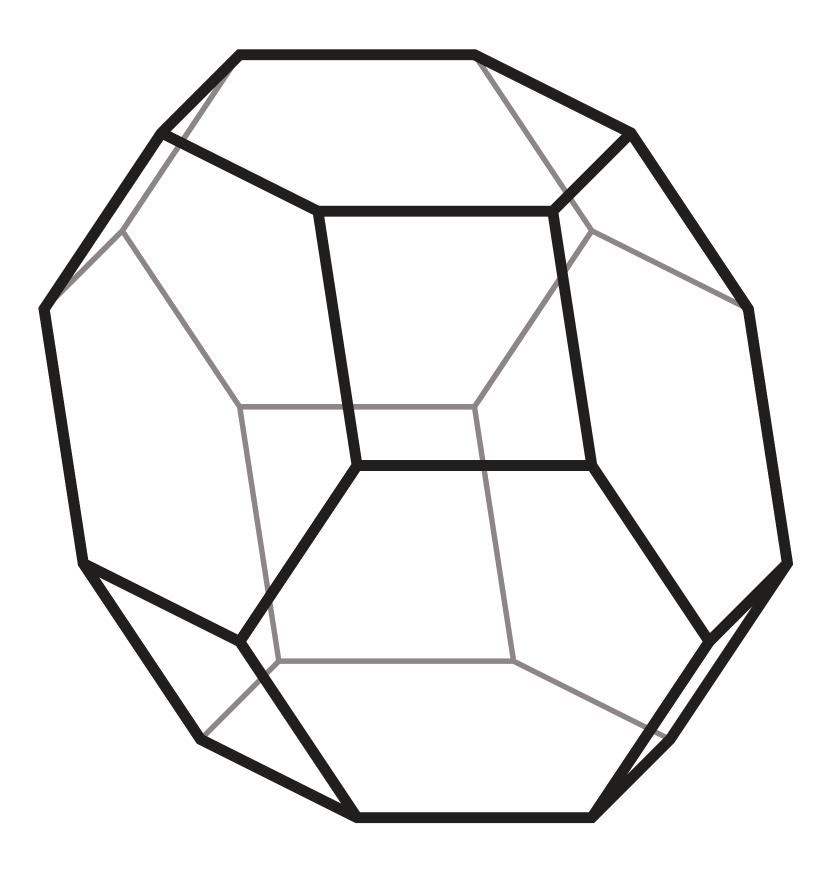




The permutahedron

One way to build it: (Schoute 1914) Gift wrap these 24 points in 4-D space: (1,2,3,4)(1,3,2,4) \ldots (4,3,2,1)and you'll get the 4-permutahedron.

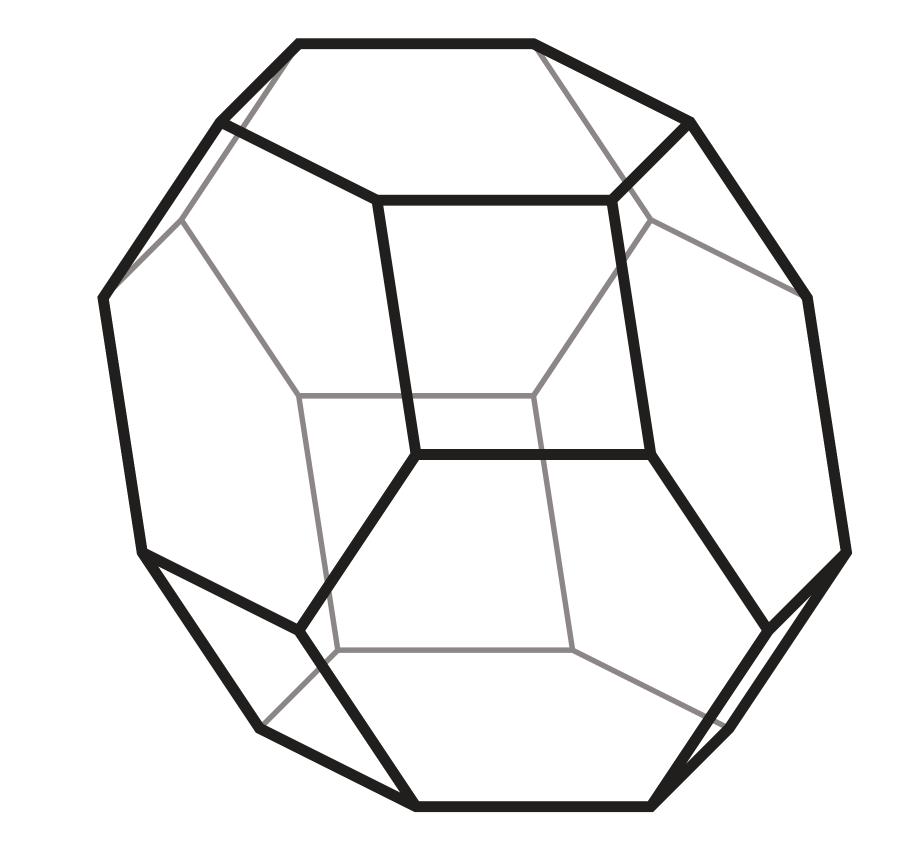
In any number of variables/dimension: The same construction works! The order of factors doesn't matter!







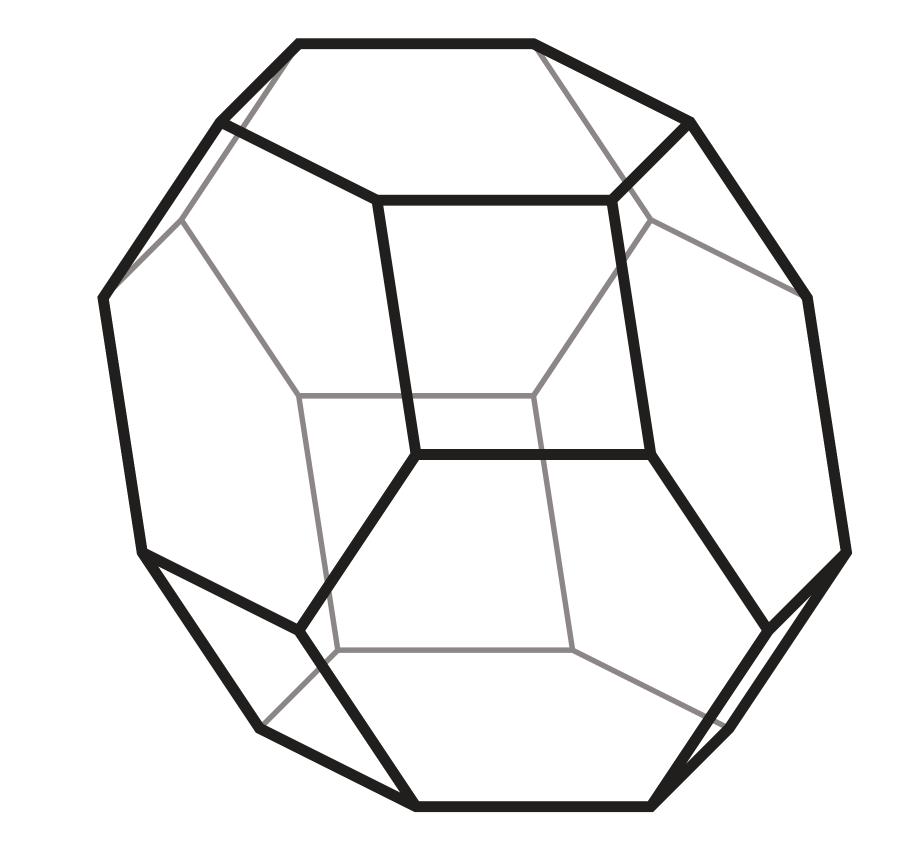
Fluorite crystal.



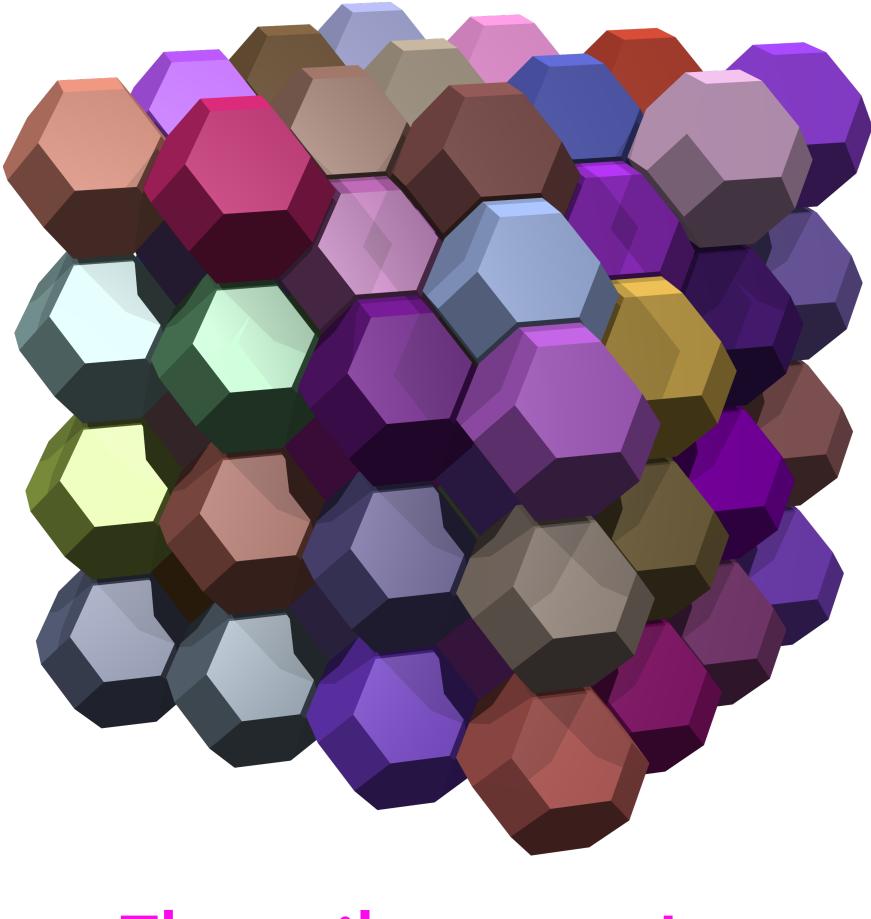




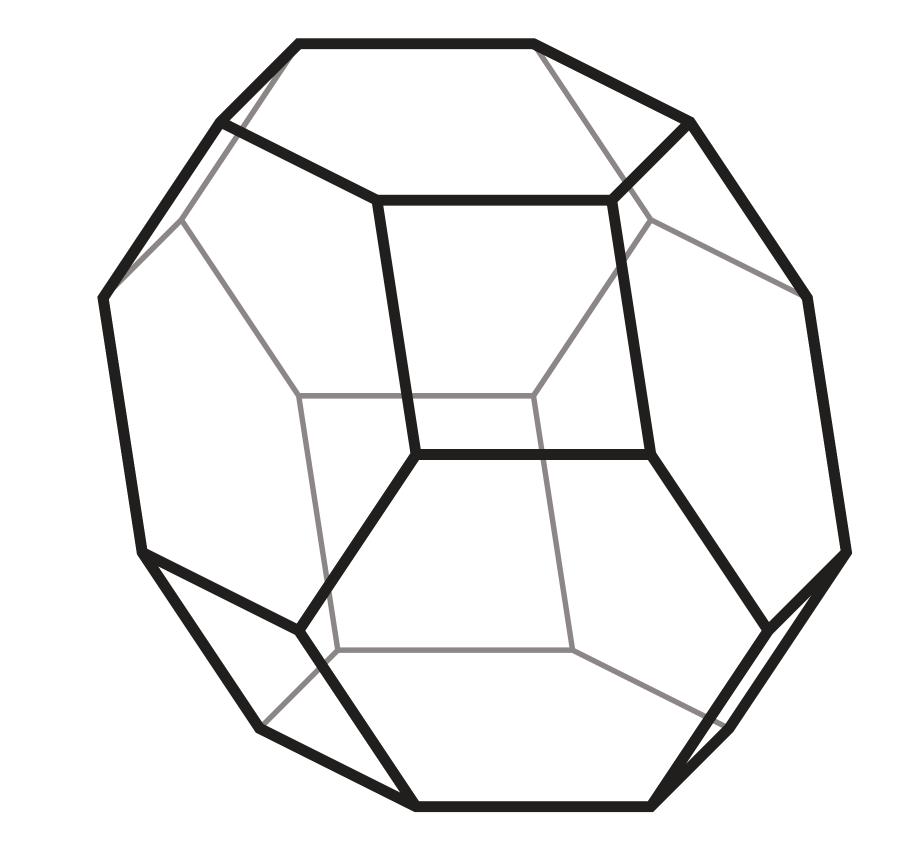
Goutte-d'Or, Paris



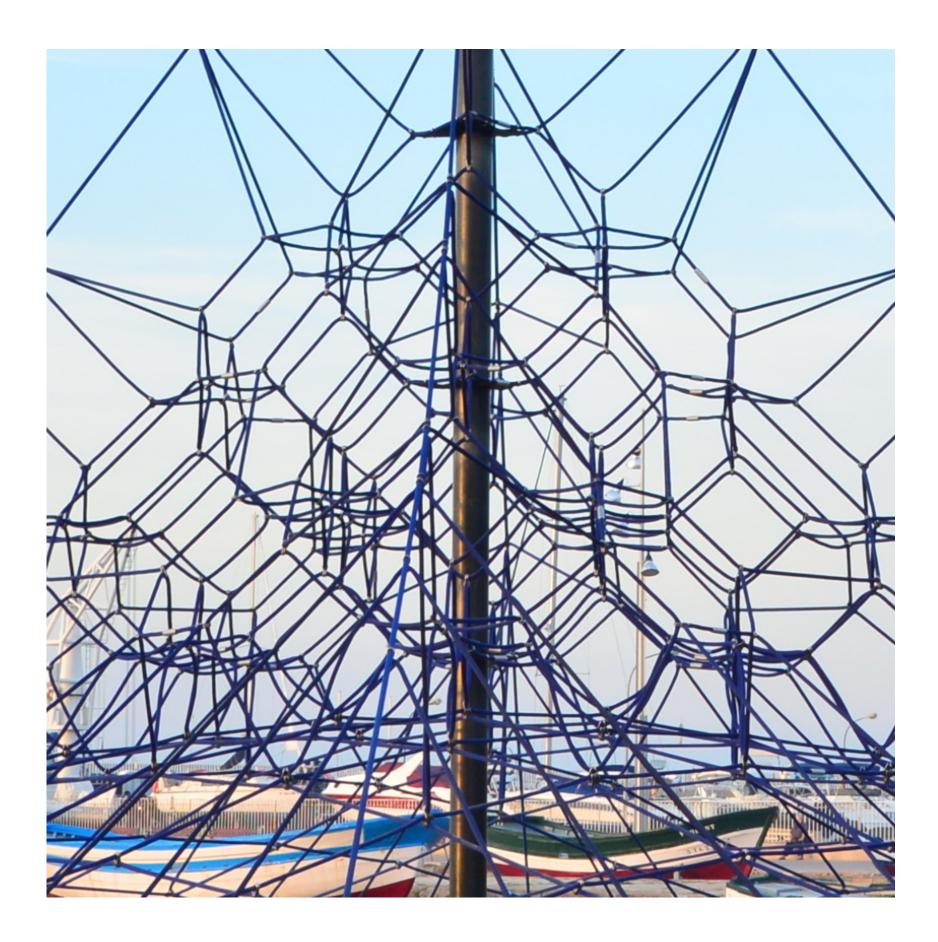




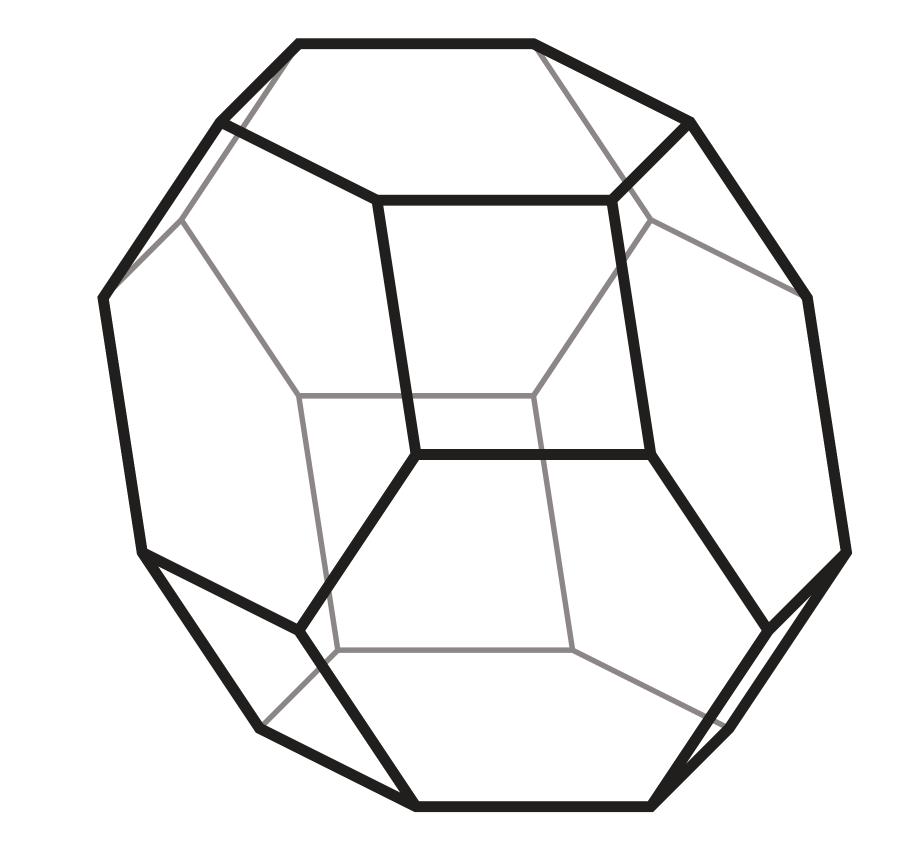
They tile space!







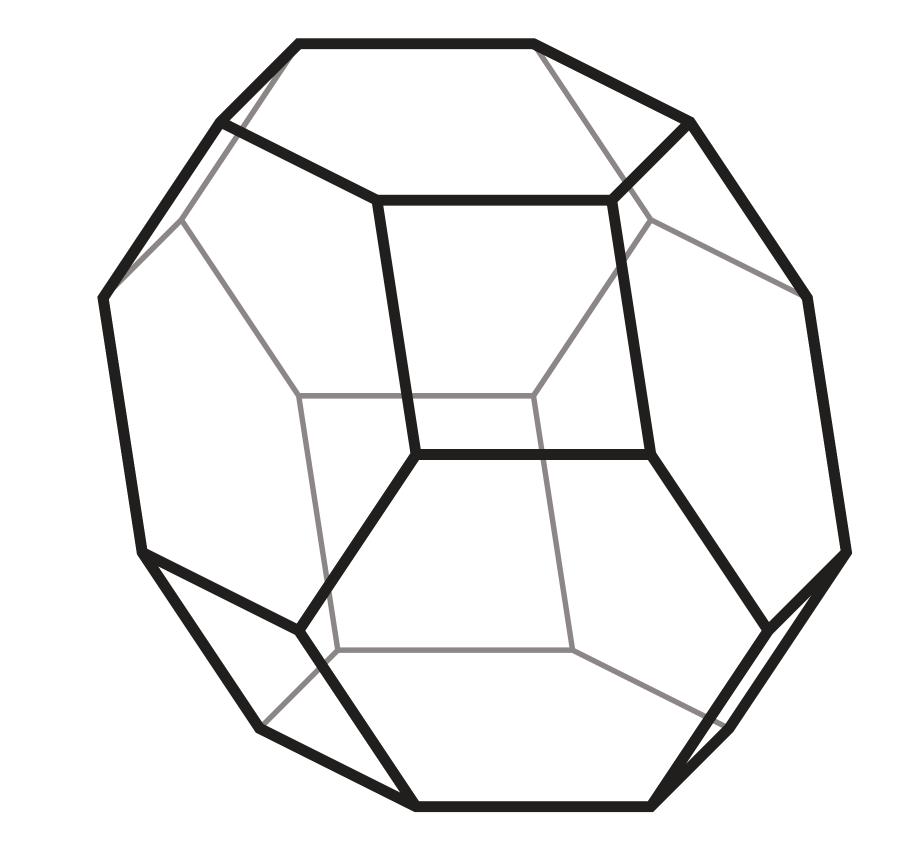
Tarragona, España





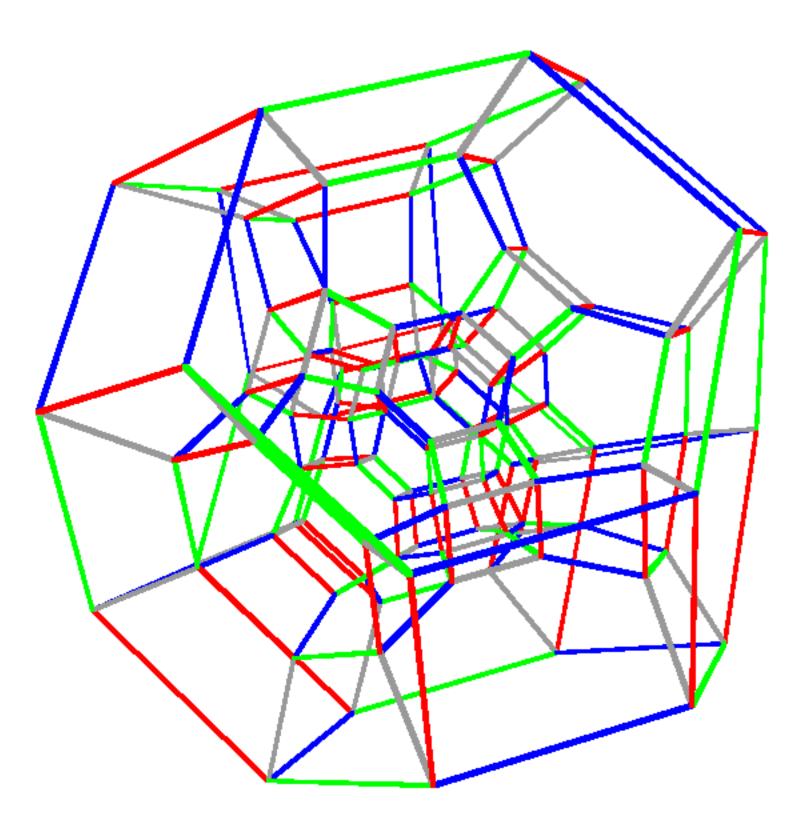


Zeolite sieve.





Order doesn't matter: more factors? The 120 products abcde, ..., edcba are equal:



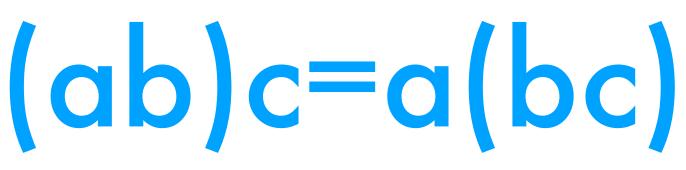
The Permutahedron The n! products of a_1, a_2, \dots, a_n are equal: The permutahedron of order n is a beautiful polyhedron. It has: dimension: n-1 vertices: n! walls: 2ⁿ - 2 volume: nⁿ⁻² The shape of n-commutativity. It tiles (n-1)-dimensional space.

4. GROUPING DOESN'T MATTER

Grouping doesn't matter: (ab)c=a(bc)

Idea: No-one can multiply 3 numbers in their head. Multiply two at a time!

Example: (a(bc))d Non-example: a(bc)d



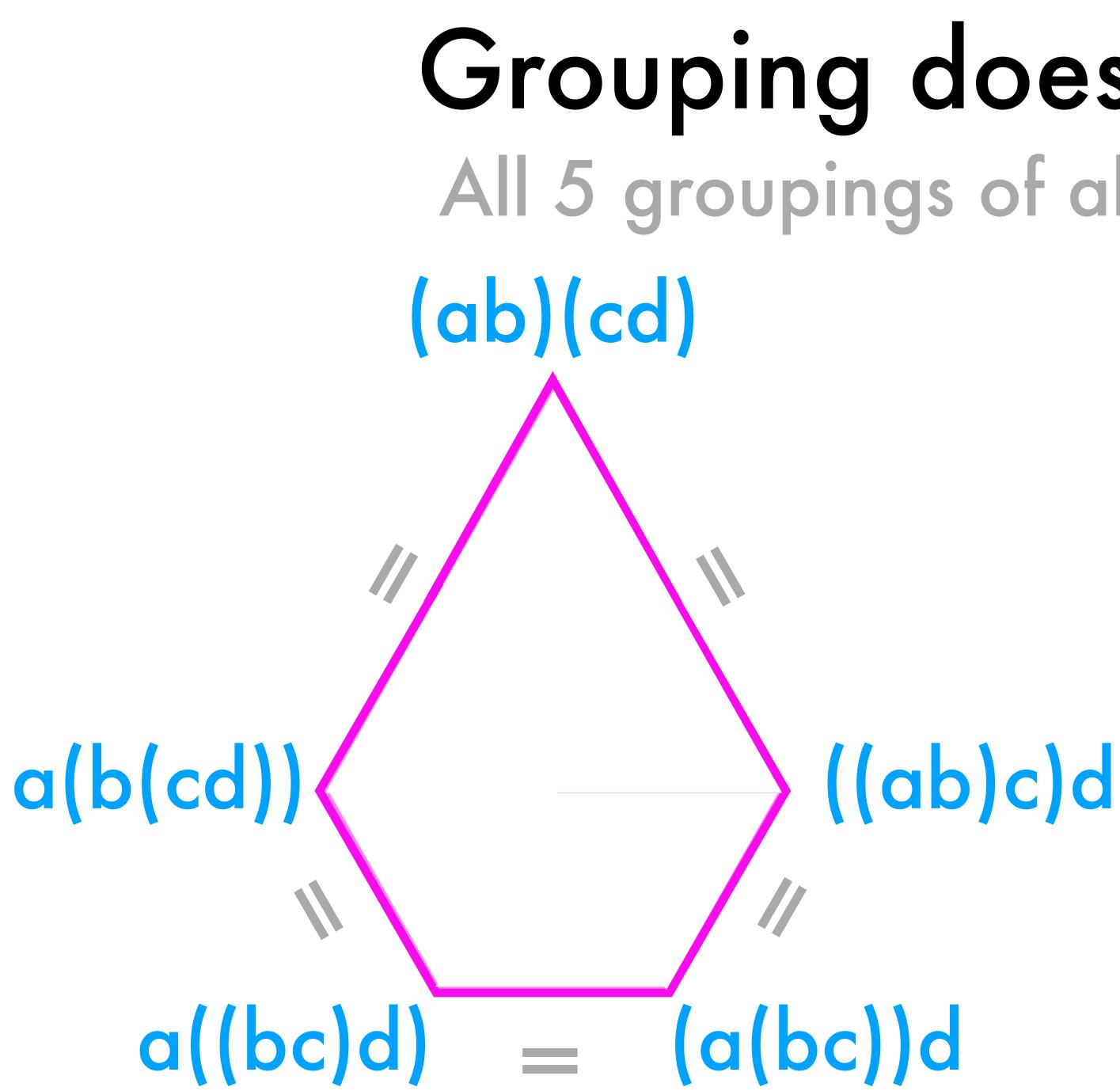
This law says that grouping doesn't matter for 3 numbers. We want to know why grouping doesn't matter for 4 numbers.

Grouping doesn't matter: (ab)c = a(bc)

Let's use the same procedure.

- 1. What are all the ways of grouping the product abcd, two factors at a time? (Without changing order of factors.)
- 2. Why do they all give the same answer? Can we prove it?

Example: (a(bc))d Non-example: a(bc)d

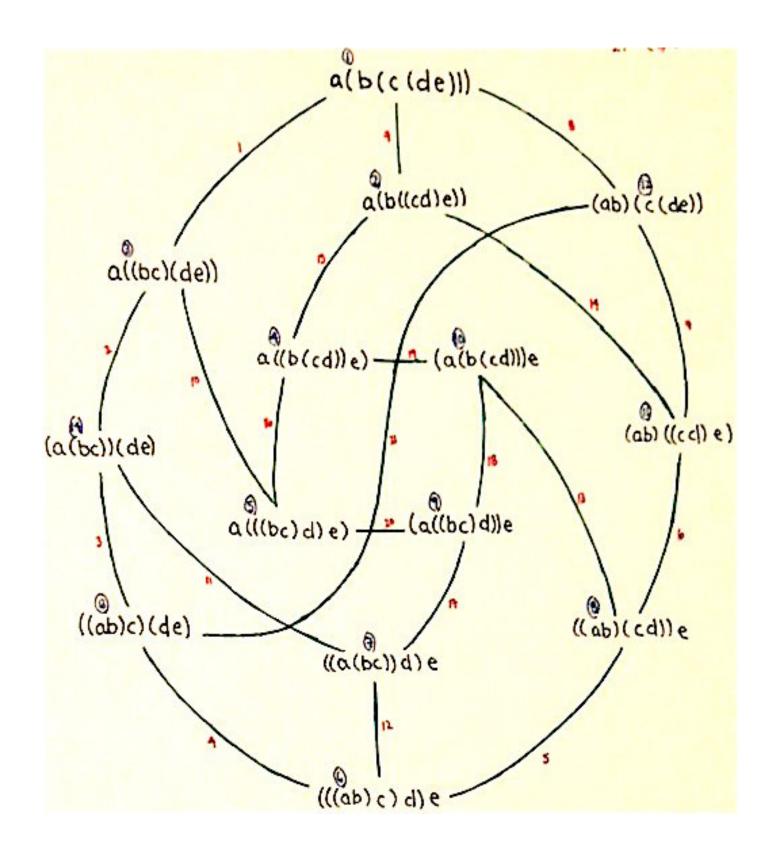


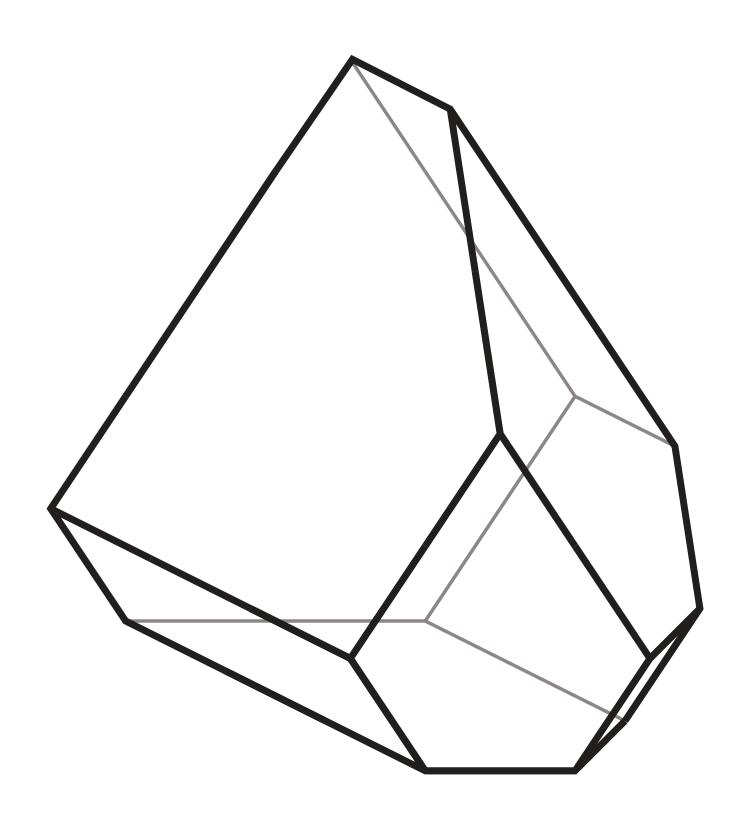
Grouping doesn't matter: All 5 groupings of abcd are equal:

The shape of 4-associativity.



Grouping doesn't matter: The 14 groupings of abcde are equal:



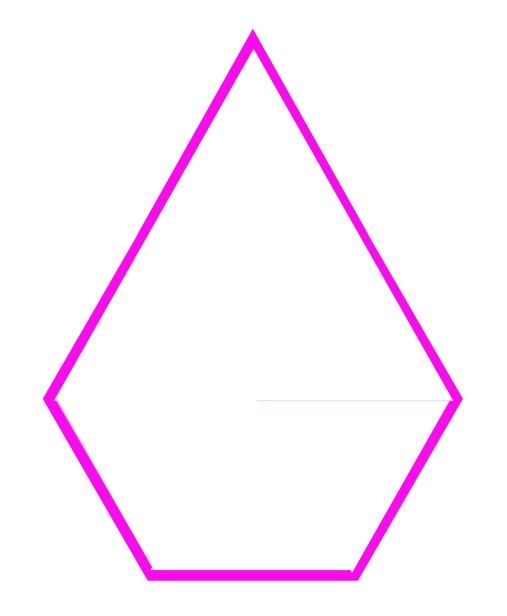


The shape of 5-associativity.

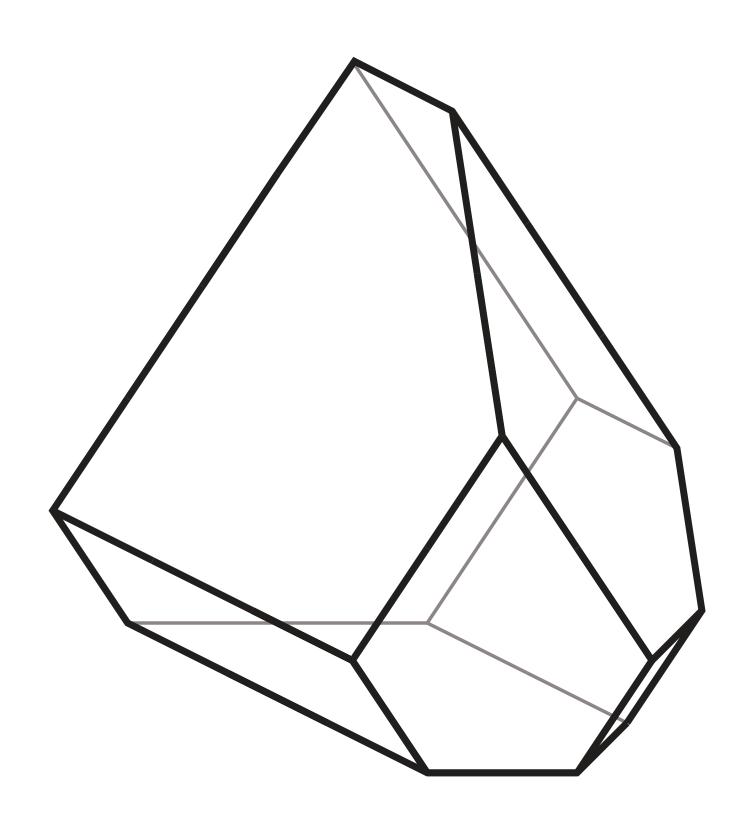


5. THE ASSOCIAHEDRON

The associahedron



The shape of 4-associativity.



The shape of 5-associativity.



The associated of the second s The n! groupings of $a_1 a_2 \dots a_n$ are equal:

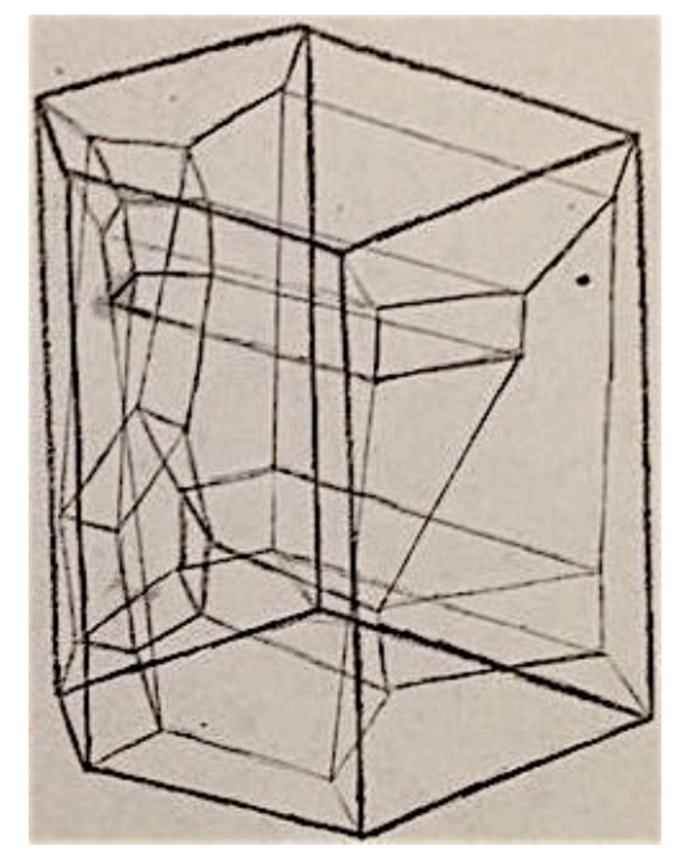
The associahedron of order n is a beautiful polyhedron. It has:

dimension: n-2

vertices: $C_n = (2n)!/n!(n+1)!$

walls: n(n+1)/2 - 1

SSS volume:



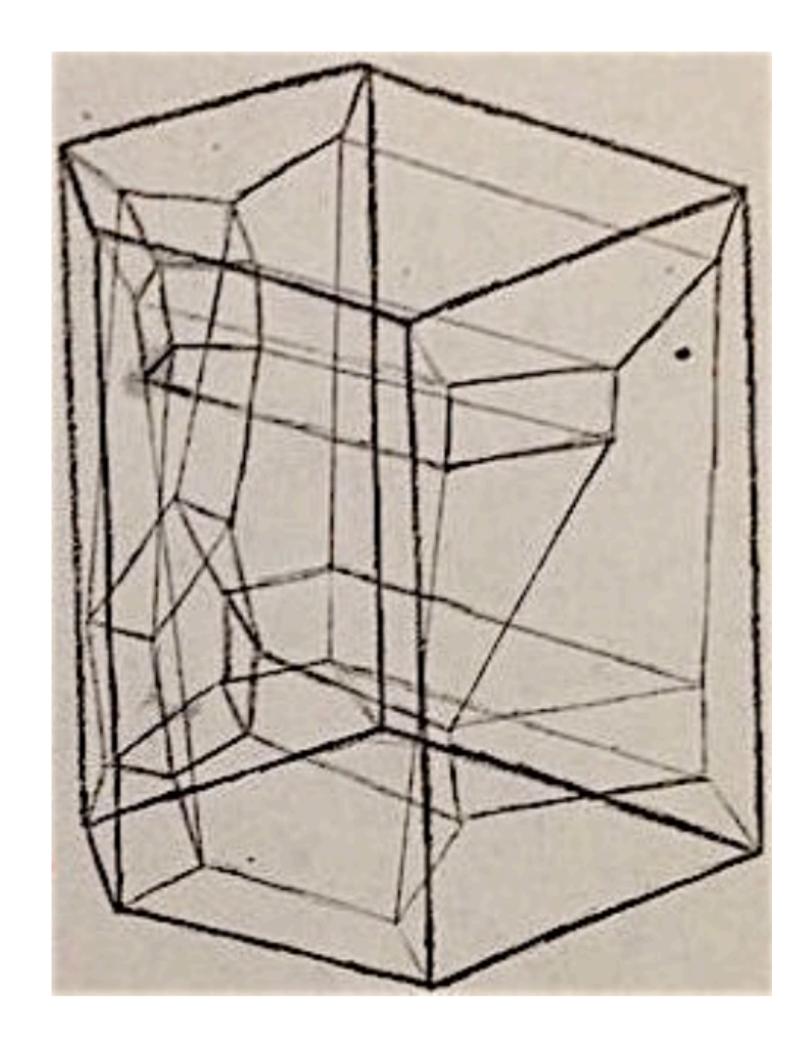
The shape of 6-associativity.



The associahedron: history

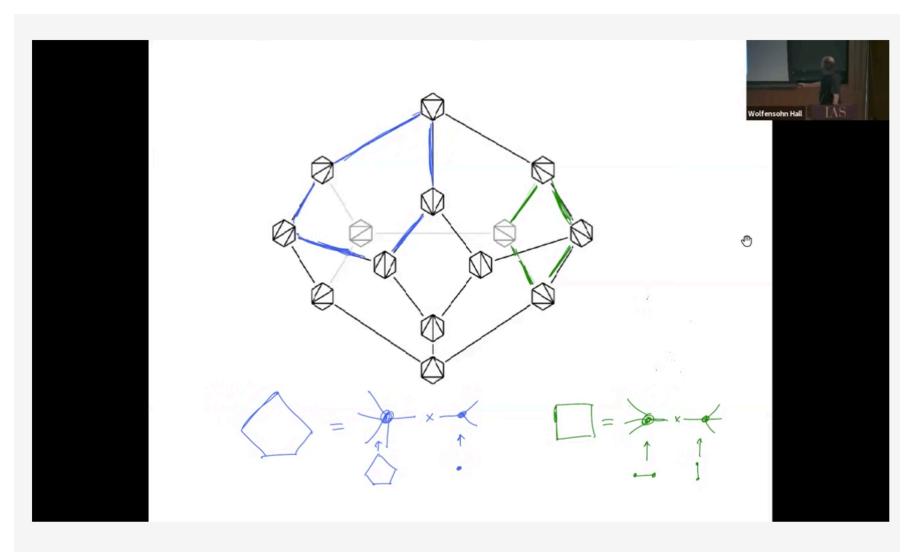
"The associahedron is a mythical polytope representing the parenthesizations of variables."

Stasheff 1963: topology Haiman 1984: geometry Loday 2004: algebra Ceballos et al 2015: combinatorics Arkani-Hamed et al 2017: physics



The shape of n-associativity.





Spacetime and Quantum Mechanics; Particles and "Strings"; Polytopes, Binary... - Nima Arkani-Hamed

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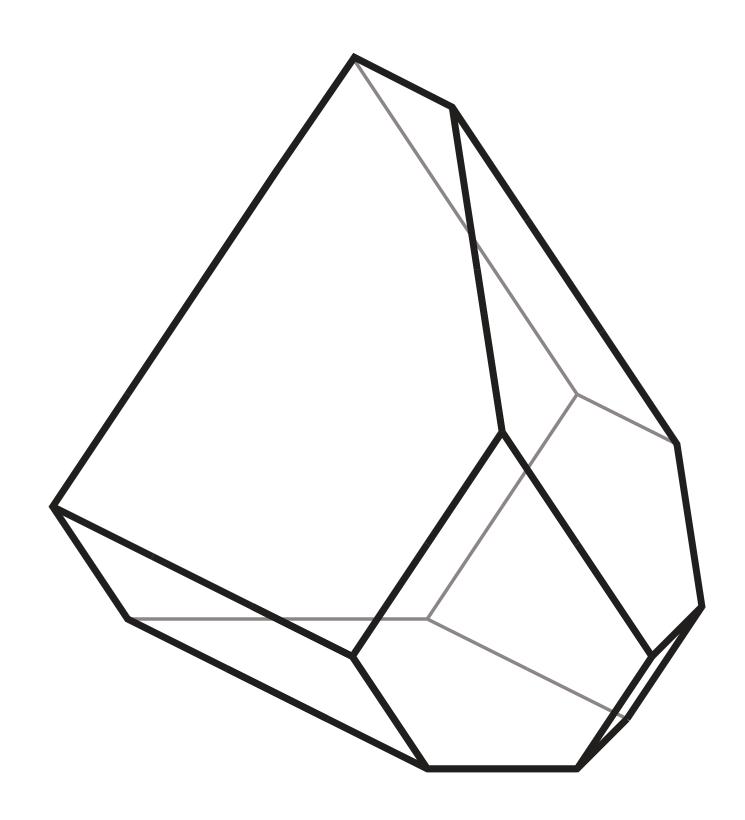
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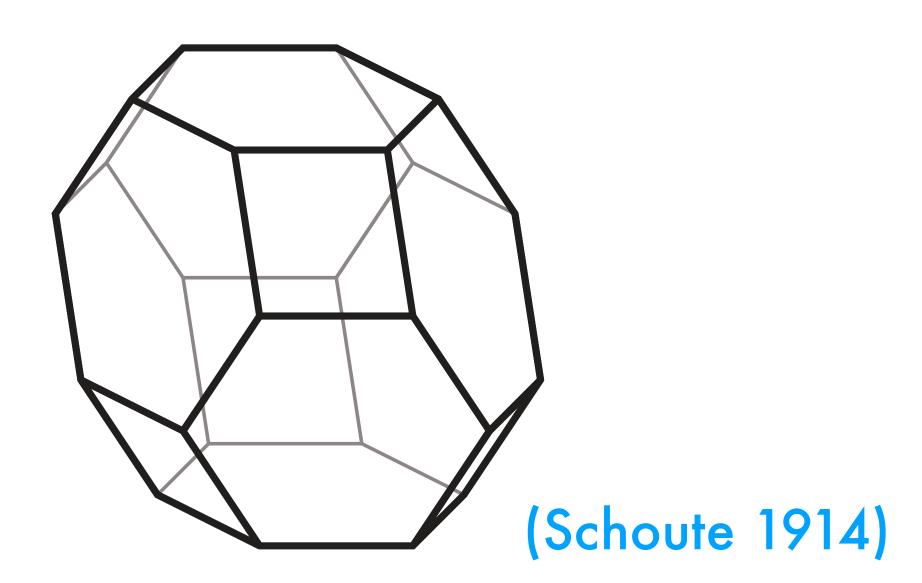
Associahedra in unexpected places



The shape of 5-associativity.



Permutahedron and associahedron, together.



The shape of 4-commutativity.

You can knock down walls of P(n) to get A(n+1)! (Postnikov 2006)

The shape of 5-associativity.



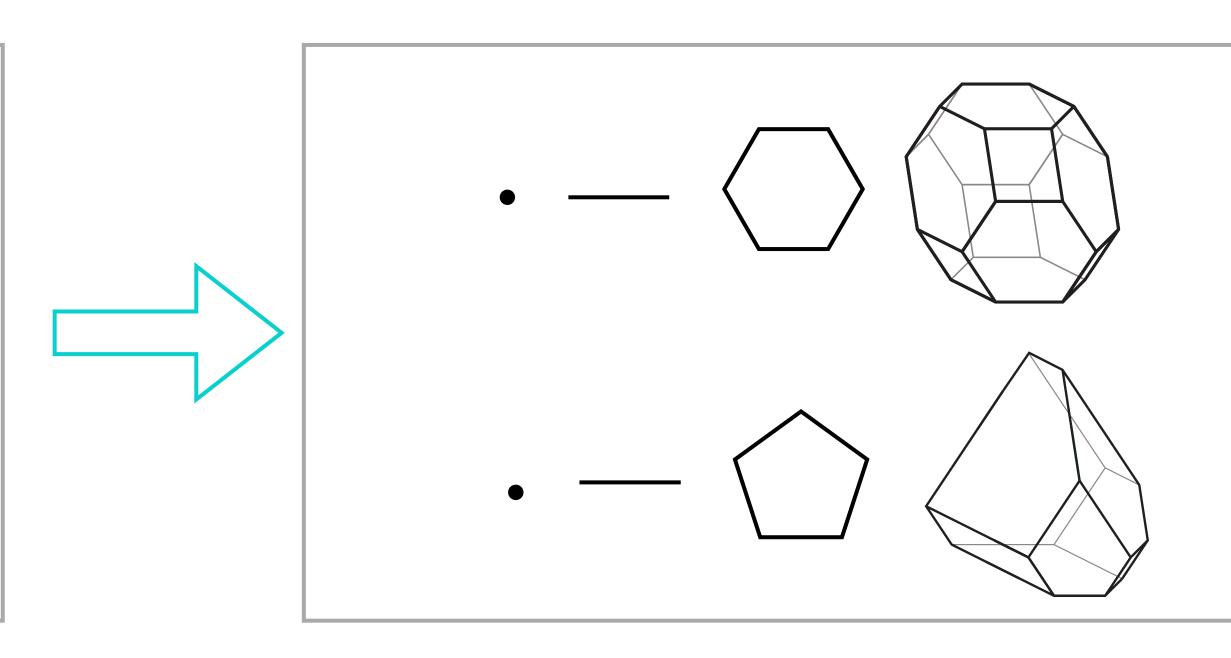


6. RECAP

$2 \times 3 \times 4 \times 5 = 120$

Different processes give the same answer!!! Who is suprised? Who is not surprised? Why?!?! This is the topic of today.

Mathematics is interconnected, and connected to other areas, in ways we can never predict. It is always worth thinking very slowly and carefully about what we think we understand!









gracias:





iji muchas gracias !!!

questions? comments? reactions?

More information:

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youtube.com/federicoelmatematico

