Marked Poset Polytopes in Representation Theory

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GT and FFLV Patterns

Consider a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$, with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$.

A *Gelfand-Tsetlin pattern for* λ (or simply a $GT(\lambda)$ pattern) is an array of integers satisfying the inequalities on the left panel of Figure 1 below.

A *Feigin-Fourier-Littelmann-Vinberg pattern for* λ (or simply a $FFLV(\lambda)$ pattern) is an array of non-negative integers such that, for any Dyck path from λ_i to λ_j, the sum of the numbers along the path is at most $\lambda_i - \lambda_j$.

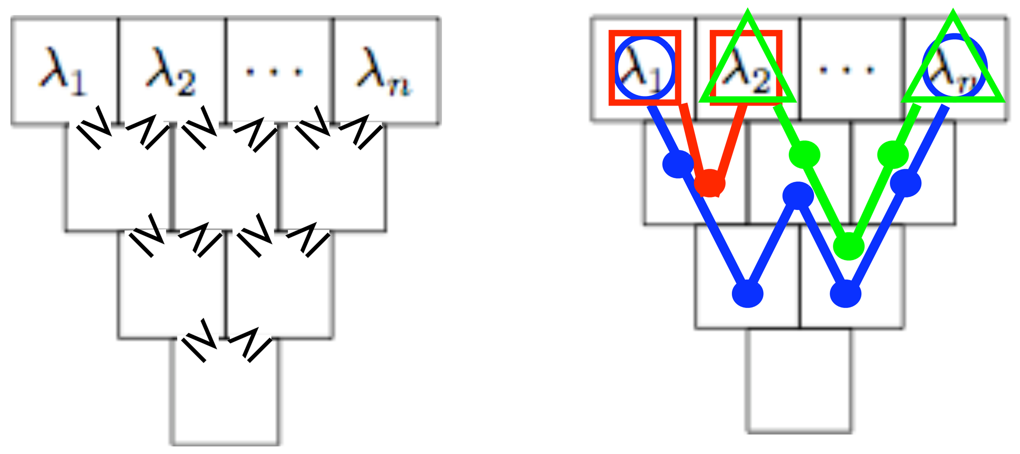


Figure 1: *Left*: GT patterns. *Right*: FFLV patterns.

Theorem 1. *(ABS '10, Feigin-Fourier-Littelmann '10) (Conjecture: Vinberg '05) The number of $GT(\lambda)$ patterns equals the number of $FFLV(\lambda)$ patterns.*

Motivation

This work is rooted in the representation theory of the special linear Lie algebra

$$\mathfrak{sl}_n(\mathbb{C}) := \{n \times n \text{ matrices of trace } 0\}, \qquad [A, B] = AB - BA.$$

The irreducible representations $V(\lambda)$ of $\mathfrak{sl}_n(\mathbb{C})$ are in bijection with partitions $\lambda = (\lambda_1, \ldots, \lambda_n)$ (modulo addition of $(1, \ldots, 1)$).

- (1950) Gelfand and Tsetlin constructed a basis for $V(\lambda)$ indexed by the GT patterns for λ. Therefore

$$\dim V(\lambda) = \text{ number of } GT(\lambda)\text{-patterns}.$$

- (2005) Vinberg proposed a conjectural construction of a basis for $V(\lambda)$ indexed by the FFLV patterns for λ.
- (2010) Feigin, Fourier, and Littelmann proved that Vinberg's conjectural basis is *independent* and *spanning*. (via two subtle algebraic arguments.) Thus

$$\dim V(\lambda) = \text{ number of } FFLV(\lambda) \text{ patterns}$$

- (2010) We found a combinatorial/discrete geometric explanation for

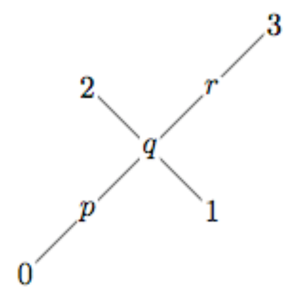
$$\text{number of } GT(\lambda)\text{-patterns} = \text{ number of } FFLV(\lambda) \text{ patterns}.$$

Marked poset polytopes

We generalize to the context of marked posets, extending work of Stanley (1986).

A *marked poset* (P, A, λ) is

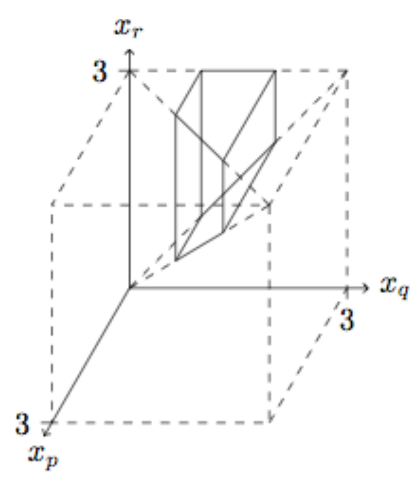
- a poset P,
- a subset $A \subseteq P$ containing all extremal elements of P, and
- a vector $\lambda \in \mathbb{Z}^P$ such that $\lambda_p \leq \lambda_q$ for $p \leq q$.



The *marked order polytope* of (P, A, λ) is

$$\mathcal{O}(P,A)_\lambda = \{x \in \mathbf{R}^{P-A} \mid \; x_p \leq x_q \text{ for } p < q, \; \lambda_a \leq x_p \text{ for } a < p, \; x_p \leq \lambda_a \text{ for } p < a\},$$

where p and q represent elements of $P - A$, and a represents an element of A.

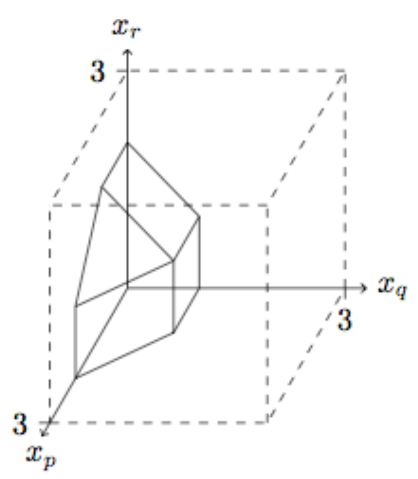


$$0 \leq x_p \leq x_q \leq x_r \leq 3$$
$$1 \leq x_q \leq 2$$

The *marked chain polytope* of (P, A, λ) is

$$\mathcal{C}(P,A)_\lambda = \{x \in \mathbf{R}^{P-A}_{\geq 0} \mid \; x_{p_1} + \cdots + x_{p_k} \leq \lambda_b - \lambda_a \text{ for } a < p_1 < \cdots < p_k < b\},$$

where a, b represent elements of A, and $p_1, \ldots, p_k$ represent elements of $P - A$.



$$x_p, x_q, x_r \geq 0$$
$$x_p + x_q + x_r \leq 3, x_q \leq 1$$
$$x_p + x_q \leq 2, x_q + x_r \leq 2$$

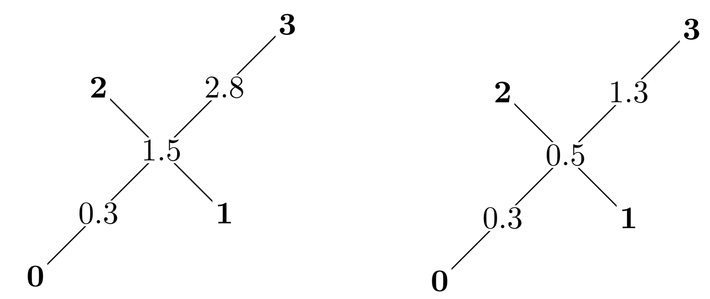
Theorem 2. *(ABS, '10) The polytopes $\mathcal{O}(P,A)_\lambda$ and $\mathcal{C}(P,A)_\lambda$ have the same Ehrhart polynomial. In particular, they have the same number of lattice points.*

They are very different combinatorially! We give a piecewise linear bijection.

The bijection

Stanley (1986) proved the analogous result for the order and chain polytopes of an unmarked poset. Essentially the same bijection works here:

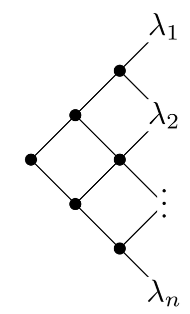
$$\phi : \mathcal{O}(P,A)_\lambda \longrightarrow \mathcal{C}(P,A)_\lambda$$
$$\phi(x)_p = x_p - \max\{x_q \mid q < p\}$$



The bijection shows that $\mathcal{C}(P,A)_\lambda$ is integral. (non-trivial!)

Back to representation theory

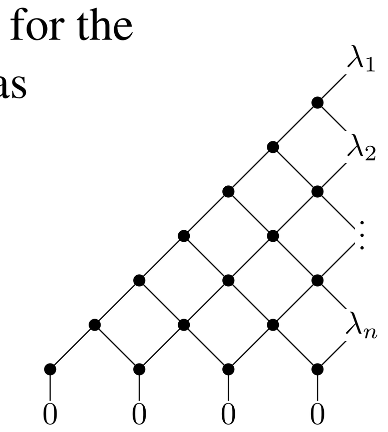
- $\mathfrak{sl}_n$: For the marked poset on the right,
  lattice points of $\mathcal{O}(P,A)_\lambda = GT(\lambda)$ patterns,
  lattice points of $\mathcal{C}(P,A)_\lambda = FFLV(\lambda)$ patterns.
  So Theorem 2 implies Theorem 1.



Question 1. *What does this bijection say about the change of basis between the GT and FFLV bases of $V(\lambda)$?*

Question 2. *The bijection induces natural subdivisions on $\mathcal{O}(P,A)_\lambda$ and $\mathcal{C}(P,A)_\lambda$. Do these have an algebraic meaning?*

- $\mathfrak{sp}_{2n+1}$: Berenstein-Zelevinsky (1989) constructed bases for the irreducible representations of all semisimple Lie algebras in terms of "generalized Gelfand-Tsetlin patterns".
  For the marked poset on the right,
  lattice points of $\mathcal{O}(P,A)_\lambda = GT(\lambda)$ patterns of $\mathfrak{sp}_{2n+1}$



Our bijection gives a definition of "FFLV patterns of $\mathfrak{sp}_{2n+1}$", and:

Theorem 3. *(Feigin-Fourier-Littelmann '11) (Conjecture: ABS '10) There is a FFLV basis for $\mathfrak{sp}_{2n+1}$ indexed by the lattice points of $\mathcal{C}(P,A)_\lambda$.*

- $\mathfrak{so}_{2n+1}$: The $GT(\lambda)$ patterns of $\mathfrak{so}_{2n+1}$ are some (non-lattice) points of $\mathcal{O}(P,A)_\lambda$ as above. This suggests:

Question 3. *Is there a FFLV basis for $\mathfrak{o}_{2n+1}$ in terms of $\mathcal{C}(P,A)_\lambda$?*

- $\mathfrak{so}_{2n}$, etc: The $GT(\lambda)$ patterns of type $D_n, E_6, E_7, E_8, F_4, G_2$ do not seem to correspond to points in a marked order polytope. What can be done here?