Measuring polytopes through their algebraic structure

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Algebra, Geometry, and Combinatorics Colloquium
December 15, 2020

F.A. and Marcelo Aguiar (Cornell) 2008-2020
Hopf monoids and generalized permutahedra
`arXiv:1709.07504`

F.A. and Mario Sanchez (Berkeley) 2020
The indicator Hopf monoid of generalized permutahedra
`arXiv:2020.11178`

What is combinatorics about? A personal view.

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We usually:

1. Study the structure of the individual objects or the set.
2. (If we like to count), use this structure to count them.

Main objective:

Understanding the underlying structure of discrete objects.

(Often this structure is algebraic, geometric, topological.)

What is alg + geom + top combinatorics about?

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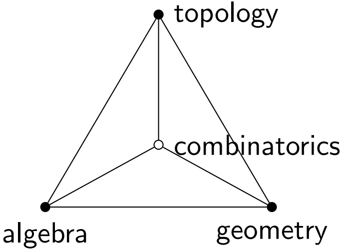

1.1. A tale of two polytopes: Permutations

$\{1, 2, \ldots, n\}$ has $n!$ permutations. How are they structured?

$n = 3$: 123, 132, 213, 231, 312, 321

What does the "space of permutations" look like?

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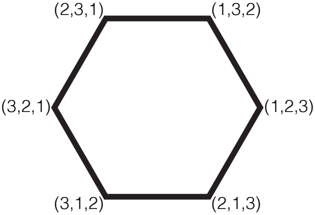

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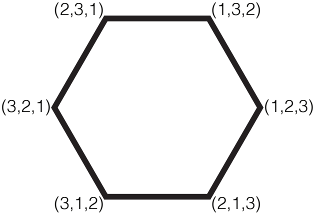

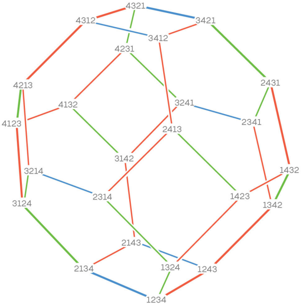

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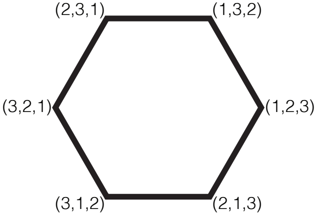

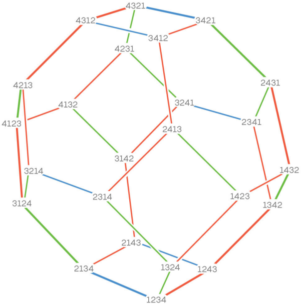

A convex polytope!

The permutahedron. Schoute 11, Bruhat/Verma 68, Stanley 80

1.2. A tale of two polytopes: Associations

$x_1 x_2 \cdots x_n$ has $\frac{1}{n+1}\binom{2n}{n}$ associations. How are they structured?

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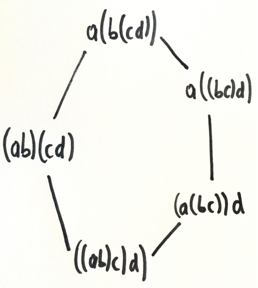

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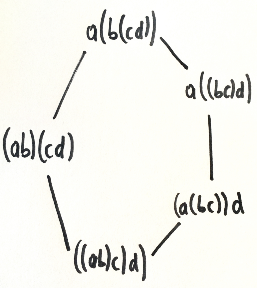

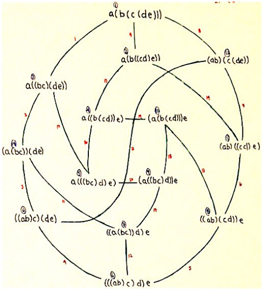

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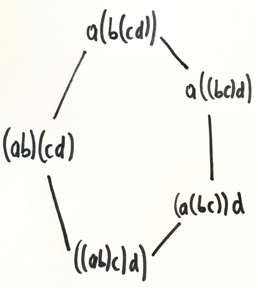

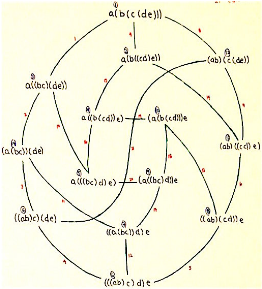

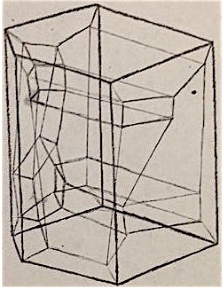

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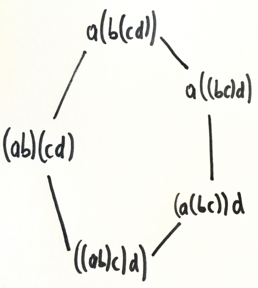
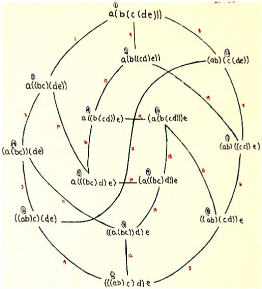
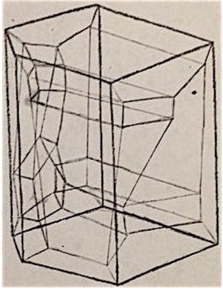


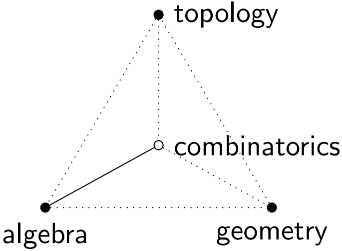




A convex polytope!

The associahedron. Stasheff 63, Haiman 84, Loday 04, Escobar 14

2. Hopf monoids.



2. Hopf monoids.

I won't assume you know what a Hopf algebra or monoid is. I didn't.

Hopf monoids refine Hopf algebras. Like categorification, they are more abstract but better suited for many combinatorial purposes.

There is a Fock functor

$$\text{Hopf monoids} \longrightarrow \text{Hopf algebras}$$

so there are Hopf algebra analogs of all of our results.

2.1. Hopf monoids: "Definition".

(Joni-Rota, *Coalgebras and bialgebras in combinatorics.*)
(Aguiar-Mahajan, *Monoidal functors, species, and Hopf algebras.*)

Think:

- A family of combinatorial structures. (graphs, posets, matroids, ...)
- Rules for "merging" and "breaking" those structures.

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A Hopf monoid $(\mathrm{H}, \mu, \Delta)$ consists of:

- For each finite set I, a vector space $\mathrm{H}[I]$.
- For each partition $I = S \sqcup T$, maps

$$\text{product} \quad \mu_{S,T} : \mathrm{H}[S] \otimes \mathrm{H}[T] \to \mathrm{H}[I]$$
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satisfying various axioms.

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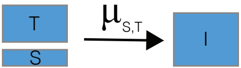
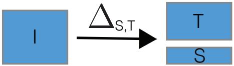
For us, $\mathrm{H}[I] = \text{span}\{\text{combinatorial structures of type } H \text{ on } I\}$

product: 

coproduct: 

2.1. Hopf monoids: Axioms.

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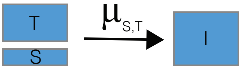
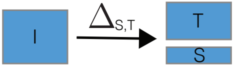
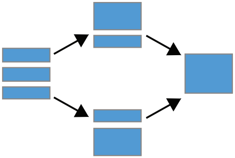
product 

coproduct 

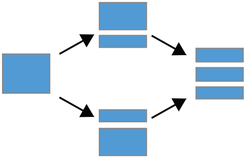
Axioms:

μ is associative.



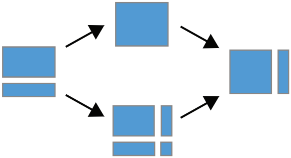
can merge several structures into one

Δ is coassociative.



can break one structure into several

μ and Δ are compatible.



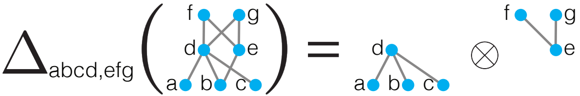
merge, then break $=$ break, then merge

Example 1: The Hopf monoid of posets.

$P[I] := \text{span}\{\text{posets on } I\}$.

Product: $p_1 \cdot p_2 = p_1 \sqcup p_2$ (disjoint union)

Coproduct: $\Delta_{S,T}(p) = \begin{cases} p|_S \otimes p|_T & \text{if } S \text{ is a lower set of } p \\ 0 & \text{otherwise} \end{cases}$



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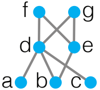
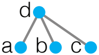
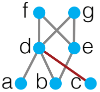
$\Delta_{abcd,efg}$ (

) = 

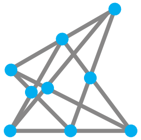
$\otimes$ 

$\Delta_{abde,cfg}$ (

) $= 0$

Example 2: The Hopf monoid of matroids.

$\mathrm{M}[I] := \mathrm{span}\{\text{matroids on } I\}$.



Matroids are a combinatorial model of independence.

They capture the properties of (linear,algebraic,graph,matching,...) independence.

Product: $m_1 \cdot m_2 \; = \; m_1 \oplus m_2$ (direct sum)

Coproduct: $\Delta_{S,T}(m) = m|_S \otimes m/_S$ where

$m|_S \; = \;$ restriction of m to S, (keep only S)

$m/_S \; = \;$ contraction of m w.r.t. S. (mod out by $\mathrm{span}(S)$)

Other Hopf monoids.

There are many interesting Hopf monoids in combinatorics, algebra, and representation theory.

A few of them:

- graphs G
- posets P
- matroids M

- set partitions Π (symmetric functions)
- paths A (Faá di Bruno)

- simplicial complexes SC
- hypergraphs HG
- building sets BS

2.2. The antipode of a Hopf monoid.

Think:	groups $\rightsquigarrow$ inverses
	Hopf monoids $\rightsquigarrow$ antipodes

(often $\mathrm{S}^2 = \mathrm{id}$)

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Takeuchi: The antipode of a connected Hopf monoid H is :

$$\mathrm{s}_I(h) = \sum_{\substack{I = S_1 \sqcup \cdots \sqcup S_k \\ k \geq 1}} (-1)^k \, \mu_{S_1,\ldots,S_k} \circ \Delta_{S_1,\ldots,S_k}(h),$$

the signed sum of all ways to (break apart then put back together).

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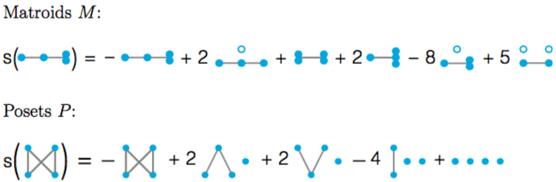
General problem.	Find the simplest possible formula for the antipode of a Hopf monoid.

(Usually there is **much** cancellation in the definition above.)

Examples: The antipode of a matroid, poset.

Ex. Takeuchi: $\mathrm{s}_I \;=\; \sum_{I=S_1\sqcup\cdots\sqcup S_k} (-1)^k\, \mu_{S_1,\ldots,S_k} \circ \Delta_{S_1,\ldots,S_k}.$

For $n = 4$ this sum has 73 terms. However,



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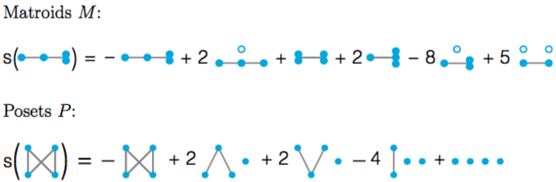

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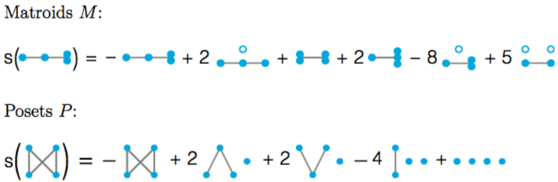


How do we explain (and predict) the simplification?

- In the past: clever combinatorics.
- Our approach: geometry + topology: Euler characteristics.

Some antipodes of interest.

There are many other Hopf monoids of interest.
Very few of their (optimal) antipodes were known.

- graphs G: ?, Humpert–Martin 10
- posets P: ?
- matroids M: ?

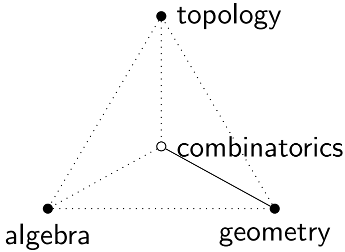
- set partitions / symm fns. Π: Aguiar–Mahajan 10
- paths A: ?

- simplicial complexes SC: Benedetti–Hallam–Michalak 16
- hypergraphs HG: ?
- building sets BS: ?

Goal: a unified approach to compute these and other antipodes.

(We do this).

3. Generalized permutahedra.



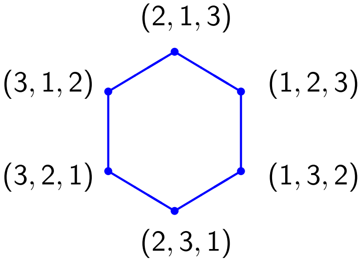
3.1. Permutahedra.

The standard permutahedron is

$$\pi_n \;:=\; \text{Convex Hull}\{\text{permutations of } \{1,2,\ldots,n\}\} \subseteq \mathbb{R}^n$$

$$\{x_1 + x_2 + x_3 = 6\}$$

$\pi_3 \;=$



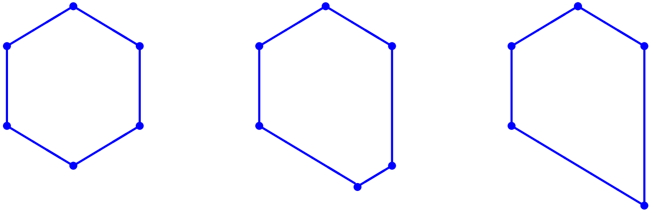
3.2. Generalized permutahedra.

Edmonds (70), Postnikov (05), Postnikov–Reiner–Williams (07),...

Equivalent formulations:

- Move the facets of the permutahedron without passing vertices.
- Move the vertices while preserving edge directions.

Generalized permutahedra:



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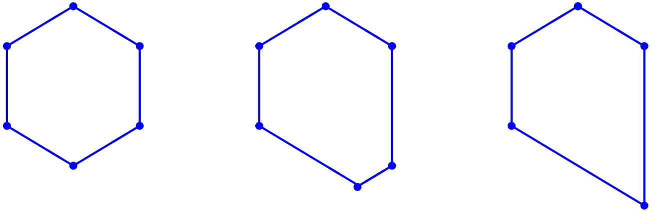
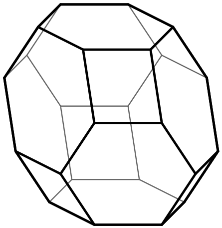


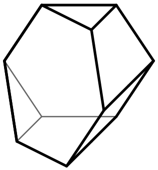
Gen. permutahedra = "polymatroids" = "submodular functions".

Many natural gen. permutahedra! Especially in optimization.

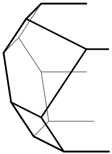
The permutahedron π_4.

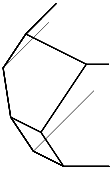


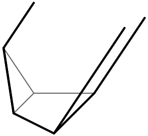
A generalized permutahedron.



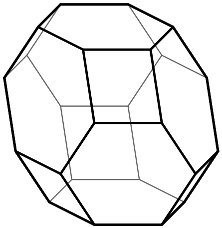
We allow unbounded ones:

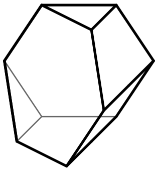




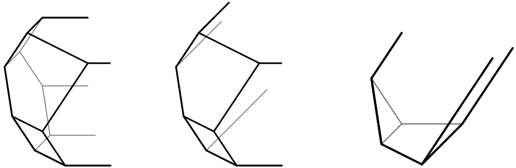


The permutahedron π_4. A generalized permutahedron.





We allow unbounded ones:



Goal. A Hopf monoid of generalized permutahedra.

How do we merge gen. permutahedra? How do we split them?

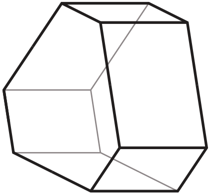
3.3. The Hopf monoid GP: Product.

Key Lemma 1. If P, Q are generalized permutahedra in $\mathbb{R}^S$ and $\mathbb{R}^T$ and $I = S \sqcup T$, then

$$P \times Q = \{(p, q) \, : \, p \in \mathbb{R}^S, q \in \mathbb{R}^T\}$$

is a generalized permutahedron in $\mathbb{R}^{S \sqcup T} = \mathbb{R}^I$.

Example: hexagon $\times$ segment =



Hopf product of P and Q:

$$P \cdot Q := P \times Q.$$

3.3. The Hopf monoid GP: Coproduct.

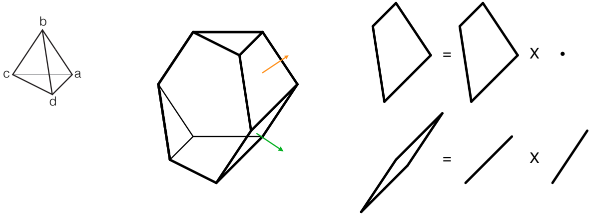
Given a polytope $P \subseteq \mathbb{R}^I$ and $I = S \sqcup T$, let

$$P_{e_{S,T}} \quad := \quad \text{face of } P \text{ where } \textstyle\sum_{s \in S} x_s \text{ is maximum.}$$

Key Lemma 2. If P is a generalized permutahedron and $I = S \sqcup T$,

$$P_{e_{S,T}} = P|_S \times P/_S$$

for generalized permutahedra $P|_S \subseteq \mathbb{R}^S$ and $P/_S \subseteq \mathbb{R}^T$.



$abcd = abd \sqcup c$

$abcd = ad \sqcup bc$

Hopf coproduct of P:

$$\Delta_{S,T}(P) := P|_S \otimes P/_S$$

3.3. The Hopf monoid of generalized permutahedra.

$\mathrm{GP}[I] :=$ span $\{$generalized permutahedra in $\mathbb{R}^I\}$.

Product: $P_1 \cdot P_2 = P_1 \times P_2$

Coproduct: $\Delta_{S,T}(P) = P|_S \otimes P/_S$

Theorem. (Aguiar–A. 08, Derksen–Fink 10)
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GP is the universal Hopf monoid of polytopes with these operations.

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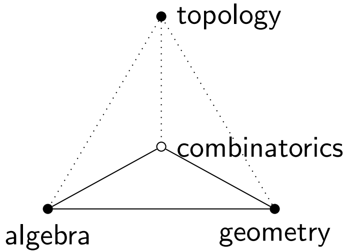

3.4. Generalized permutahedra: Posets, matroids.

There is a long tradition of modeling combinatorics geometrically. There are polyhedral models:

P poset $\rightarrow$ poset cone C_P (Geissinger 81)

$C_P : \text{cone}\{e_i - e_j \,:\, i < j \text{ in } P\}.$

M matroid $\rightarrow$ matroid polytope P_M (Edmonds 70, GGMS 87)

$P_M = \text{conv}\{e_{i_1} + \cdots + e_{i_k} \mid \{i_1, \ldots, i_k\} \text{ is a basis of } M\}.$

Proposition. (Aguiar–A. 08)
These maps are inclusions of Hopf monoids!

$$\mathrm{M} \hookrightarrow \mathrm{GP}, \qquad \mathrm{P} \hookrightarrow \mathrm{GP}.$$

(Similarly for graphs, simplicial complexes, paths, building sets,...)

3.5. The antipode of GP.

Theorem. (Aguiar–A.) Let P be a generalized permutahedron.

$$\mathrm{s}(P) \; = \; \sum_{Q \leq P} (-1)^{\operatorname{codim} Q} \, Q.$$

The sum is over all faces Q of P.

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Proof. Takeuchi:

$$\begin{aligned}
\mathrm{s}(P) \quad &= \sum_{I = S_1 \sqcup \cdots \sqcup S_k} (-1)^k \, \mu_{S_1,\ldots,S_k} \otimes \Delta_{S_1,\ldots,S_k}(P) \\
&= \sum_{I = S_1 \sqcup \cdots \sqcup S_k} (-1)^k \, P_{S_1,\ldots,S_k}
\end{aligned}$$

where $P_{S_1,\ldots,S_k}$ = face of P in direction $S_1 | \cdots | S_k$.

Coeff. of a face Q: huge sum of 1s and -1s. How to simplify it?

It is the reduced **Euler characteristic** of a sphere! $\square$

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Coeff. of a face Q: huge sum of 1s and -1s. How to simplify it?

It is the reduced **Euler characteristic** of a sphere! □

This is the best possible formula. No cancellation or grouping.

(One advantage of working with Hopf monoids!)

3.5. The antipode of GP.

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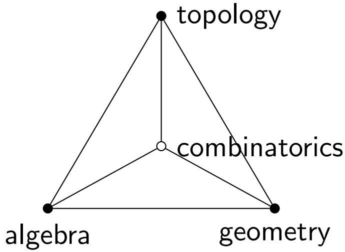


The antipodes of matroids, posets.

For Hopf algebras:

Matroids:

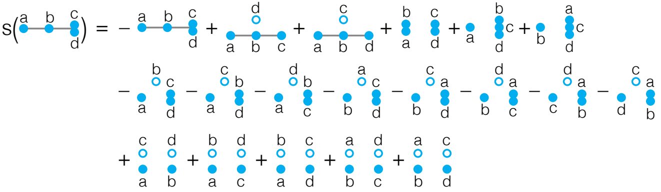
Posets:

What are these numbers??

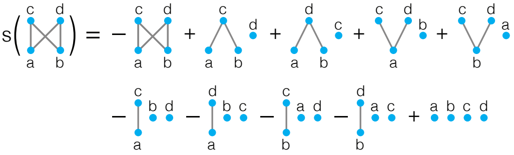
The antipodes of matroids, posets.

For Hopf monoids: (“Categorify”!)

Matroids:

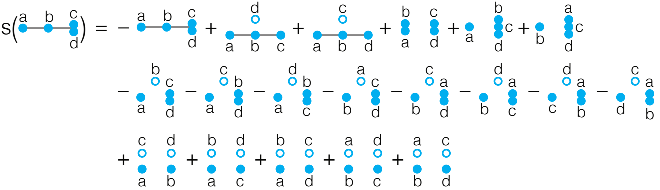


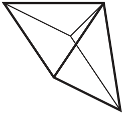
Posets:



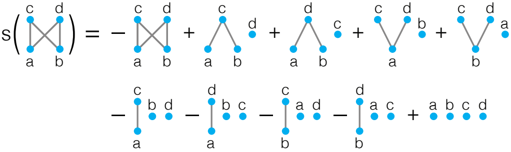
The antipodes of matroids, posets.

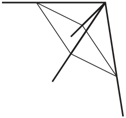
Matroids:



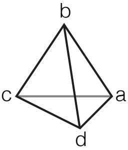


Posets:





$$\mathrm{s}(P) = \sum_{Q\leq P} (-1)^{\operatorname{codim} Q}\, Q.$$



Many antipode formulas.

Theorem. (Aguiar–A.) Let P be a generalized permutahedron.

$$\mathrm{S}(P) = \sum_{Q \leq P} (-1)^{\operatorname{codim} Q}\, Q.$$

objects	polytopes	Hopf algebra	antipode
set partitions	permutahedra	Joni-Rota	Joni-Rota
paths	associahedra	Joni-Rota, new	Haiman-Schmitt, new
graphs	graphic zonotopes	Schmitt	new, Humpert-Martin
matroids	matroid polytopes	Schmitt	new
posets	poset cones	Schmitt	new
submodular fns	polymatroids	Derksen-Fink, new	new
hypergraphs	hg-polytopes	new	new
simplicial cxes	new: sc-polytopes	Benedetti et al	Benedetti et al
building sets	nestohedra	new, Grujić et al	new
simple graphs	graph associahedra	new	new

Lots of interesting algebra and combinatorics. **Questions?**

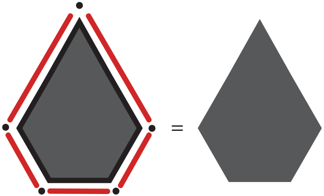
4.1. Two intriguing observations: 1

If we could actually add and subtract polytopes, the antipode would be the Euler involution of McMullen:

$$\mathrm{s}(P) \quad = \quad \sum_{Q \leq P} (-1)^{\operatorname{codim} Q}\, Q$$

$$\text{“}=\text{''} \quad (-1)^{\operatorname{codim} P}\, \operatorname{interior}(P)$$

Example: $s($  $) =$



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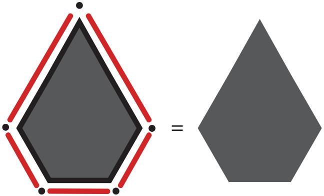

$$\mathbb{1}_P(x) = \begin{cases} 1 & \text{if } x \in P \\ 0 & \text{if } x \notin P \end{cases}$$

To make this work, instead of P, can we use indicator function $\mathbb{1}_P$?

4.1. Two intriguing observations: 2

Many combinatorial invariants are also polytopal "measures"!

Example: For posets, $f(P) = \sum_{A \text{ antichain}} t^{|A|}$ antichain: no $<$ relations

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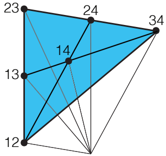

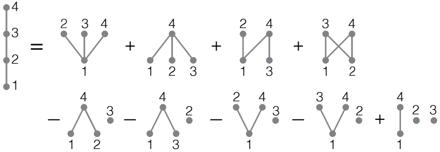

$$1 + 4t = (1 + 4t + 3t^2 + t^3) + (1 + 4t + 3t^2 + t^3) + (1 + 4t + 3t^2) + (1 + 4t + 2t^2)$$

$$-(1 + 4t + 4t^2 + t^3) - (1 + 4t + 4t^2 + t^3) - (1 + 4t + 4t^2 + t^3) - (1 + 4t + 4t^2 + t^3) + (1 + 4t + 5t^2 + 2t^3)$$

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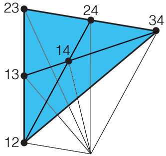

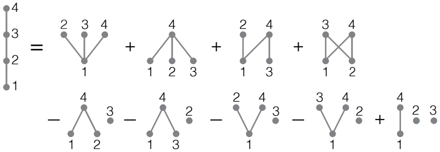

$$1 + 4t = (1 + 4t + 3t^2 + t^3) + (1 + 4t + 3t^2 + t^3) + (1 + 4t + 3t^2) + (1 + 4t + 2t^2)$$
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They are valuations: If $P_1, \ldots, P_k$ subdivide P, then

$$f(P) = \sum_{i=1}^{k} (-1)^{\dim P - \dim P_i} f(P_i).$$

Note: $f(P) = \mathbb{1}_P$ satisfies this!

4.2. A Hopf theoretic explanation

The inclusion-exclusion subspaces:

$$\mathbf{ie} := \operatorname{span}\left\{ P - \sum_i (-1)^{\dim P - \dim P_i} P_i \,\middle|\, \{P_i\} \text{ subdivides } P \right\} \subset \mathbf{GP}.$$

and the quotient

$$\begin{aligned} \mathbb{I}(\mathbf{GP}) &:= \operatorname{span}\{\mathbb{1}_P \mid P \text{ is a generalized permutahedron in } \mathbb{R}^I\} \\ &\cong \mathbf{GP}/\mathbf{ie}, \end{aligned}$$

Theorem. (A.-Sanchez 20) The Hopf monoid GP descends to $\mathbb{I}(GP)$.

Analogs shown by Derksen-Fink 10 and Bastidas 20.

Corollary. (A.-Sanchez 20)
- The antipode of $\mathbb{I}(GP)$ **is** $\mathrm{S}(P) = (-1)^{\operatorname{codim} P}\, \text{interior}(P)$.
- Invariants that come from Hopf theory **are** valuations!

4.3. Applications

We get a method to easily discover/prove that these are valuations:

For matroids:

Valuative invariant	
Chow class in permutahedral variety	(Fulton, Sturmfels)
Chern-Schwartz-MacPherson cycles	(Lopez, Rincon, Shaw)
volume polynomial	(Eur)
Kazhdan-Lusztig polynomial	(Elias, Proudfoot, Wakefield)
motivic zeta function	(Jensen, Kutler, Usatine)
universal invariant	(Derksen-Fink)
Tutte polynomial	(Speyer)

For posets:

Valuative invariant	
order polynomial	(Stanley)
Tutte polynomial	(Gordon)
antichain polynomial	
order ideal polynomial	
Poincaré polynomial	(Dorpalen-Barry, Kim, Reiner)

Questions?

4.4. Why care about valuations $\leftrightarrow$ subdivisions?

Matroid subdivisions:

Ways of cutting a matroid polytope into smaller ones. Contexts:

compactifying moduli space of hyperplane arrs.	Kapranov
compactying Schubert cells in the Grassmannian	Lafforgue
"linear spaces" in tropical geometry	Speyer, Ardila-Klivans

Use valuations to measure the complexity of matroid subdivisions!

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Use valuations to measure the complexity of matroid subdivisions!

Poset subdivisions:

Ways of cutting a (poset cone)/(root polytope) into smaller ones.

maximal minors of matrices	Bernstein-Zelevinsky, Babson-Billera
quasi-classical Yang-Baxter algebra	Kirillov, Mészáros
$(x_{ij}x_{jk} = x_{ik}x_{ij} + x_{jk}x_{ik} + \beta x_{ik}, x_{ij}x_{kl} = x_{kl}x_{ij})$	

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Use valuations to measure the complexity of poset subdivisions?

Building set subdivisions:

Theorem. (A.-Sanchez 20) There are no nestohedral subdivisions.

4.5. Universality

Question:
Why do generalized permutahedra come up so much in this theory?

Character on a Hopf monoid H:
Multiplicative function $\zeta : H \to R$: $\zeta(h)\zeta(h') = \zeta(h \cdot h')$

Example: On GP, $\beta(P) = \begin{cases} (-1)^{|I|} t^p & \text{if } P \text{ is bounded, on hyperplane } \sum_{i \in I} x_i = p \text{ in } \mathbb{R}^I, \\ 0 & \text{if } P \text{ is unbounded.} \end{cases}$

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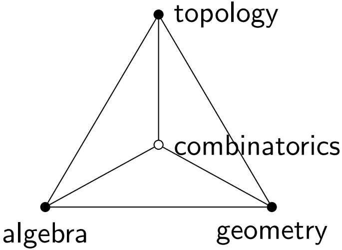
Theorem. (A.-Sanchez 20) $(\mathbb{I}(GP), \beta)$ is the terminal object in the category of Hopf monoids with polynomial characters.

Any Hopf monoid H with character $\zeta : H \to \mathbb{F}[t]$ factors through $\mathbb{I}(GP)$:
There's a (unique) Hopf morphism $\hat{\zeta} : H \to \mathbb{I}(GP)$ such that $\beta \circ \hat{\zeta} = \zeta$.

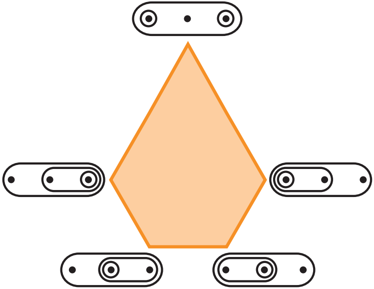
So: Every polynomial character passes through gen. permutahedra!

What is alg + geom + top combinatorics about?

Understanding the underlying structure of discrete objects.
(Often this structure is algebraic, geometric, topological.)



¡Muchas gracias!



Federico Ardila and Marcelo Aguiar
Hopf monoids and generalized permutahedra
`arXiv:1709.07504`

Federico Ardila and Mario Sanchez
The indicator Hopf monoid of generalized permutahedra
`arXiv:2020.11178`