Polytopes Lecture Index

Lecture 1

The combinatorics of 2-D polytopes is boring. For 3-D polytopes it is more interesting. (Euler's theorem, Steinitz's theorem.) For 4-D polytopes it is much more difficult and not fully understood.

Lecture 2

The algebraic definition of a polytope matches our geometric intuition. The polytope determined by a set of points is the smallest convex set containing them. Some examples of polytopes: simplices, cubes, crosspolytopes.

Lecture 3

One can build new polytopes from old ones using intersections, Minkowski sums, and products. The main theorem for polytopes says that polytopes can be described in terms of convex hulls (V-description), or inequalitites (H-description). We prove one direction of this. (V $\Rightarrow$ H)

Lecture 4

We sketch the proof of the second direction of the main theorem for polytopes. ($H \Rightarrow V$)

Lecture 5

After a brief discussion of polarity, we prove Caratheodory's theorem and one version of the Farkas lemma.

Lecture 6

We conclude the discussion of Farkas's lemmas.

Lecture 7

We formally define the face of a polytope. After a brief discussion of affine spaces, we look at examples of f-vectors and f-polynomials of polytopes. We prove that a polytope is the convex hull of its vertices and some other "obvious" facts about polytopes.

Lecture 8

One can build new polytopes from old ones by takings pyramids and vertex figures. We define the face lattice of a polytope. Two polytopes are combinatorially equivalent if they have isomorphic face lattices.

Lecture 9

The polar of a polytope is formally defined. We prove some key facts about polar polytopes.

Lecture 10

We see why the face poset of the polar of P is the opposite to the face poset of P. We discuss simple and simplicial polytopes, and observe that these are "dual" concepts. We begin to discuss cyclic polytopes.

Lecture 11

We complete the combinatorial description of the cyclic polytope and prove that it is $d/2$-neighborly.

Lecture 12

We begin to discuss graphs of polytopes and linear programming.

Lecture 13

Applications of linear programming: the minimal spanning tree problem with cost function c is equivalent to finding the minimal face of the the spanning tree polytope in direction c.

Lecture 14
The Hirsch conjecture says that the maximum diameter of a d-polytope with n facets is at most $n-d$. We provide an outline for Francisco Santos' proof that the Hirsch conjecture is false.

Lecture 15
For simple polytopes, the graph of P determines P.

Lecture 16
Any polytope P has a triangulation. We prove this proposition via regular subdivisions.

Lecture 17
We continue to explore triangulations of polytopes. In particular, the number of triangulations of an $(n+2)$-gon is the catalan number C_n.

Lecture 18
Guest lecture by Tristram Bogart. We characterize the triangulations of the crosspolytope of dimension d.

Lecture 19
Counting lattice points in polytopes. We compute the Ehrhart polynomial, interior Ehrhart polynomial, and Ehrhart series of the d-cube, and the d-simplex.

Lecture 20
Counting lattice points in polytopes through partition functions and representation theory of Lie algebras.

Lecture 21
Generating functions for cones.

Lecture 22
We compute the generating function of a polytope P via the generating function of cone(P).

Lecture 23
We prove Stanley's non-negativity theorem, and discuss the Erhart series and polynomial for a lattice d-polytope P in terms of the (non-negative) h-vector of P.

Lecture 24
Ehrhart reciprocity says that the evaluation of the Ehrhart polynomial of a d-polytope P at $-t$ is equal to the number of lattice points in the interior of P times a factor of -1^d. The proof of a similar result, Stanley's reciprocity with regards to generating

Lecture 25
We explore Ehrhart theory for rational polytopes. If P is a rational d- polytope, then $L_P(n)$ is a quasipolynomial. A nice application is Pick's theorem, which gives the area of a lattice polygon in terms of the number of interior lattice points and the number of boundary lattice points.

Lecture 26

We look more at f-vectors of a polytope. We prove Euler's relation for a d-polytope P and define the h-vector of P in terms of the f-vector. The h-vector of a simplicial polytope is symmetric.

Lecture 27
The formula for the d-volume of a pyramid with base B and height h is given. We also look at the volume of a minkowski sum of two polytopes in terms of the mixed volumes of the two polytopes.

Lecture 28
We prove that the d-volume of a minkowski sum of polytopes, $\mathrm{vol}_d(rP+sQ)$, is a homogeneous polynomial of degree d in r and s. This has a nice application to algebraic geometry for counting the number of solutions to a system of polynomial equations.

Lecture 29
Bernstein's theorem gives the number of isolated solutions to a system of polynomial equations in terms of the mixed volume of the Newton polygons of each polynomial. As a corollary we discuss the volume of the permutahedron.

Lecture 30
We continue the discussion of the volume of the permutahedron Π_n, which is equal to the number of spanning trees of the complete graph K_n.

Lecture 31
The volume of the Pittman-Stanley Polytope is discussed in terms of reverse parking functions.

Lecture 32
Conclusion of the volume of the Pittman-Stanley polytope and parking functions. There are exactly $(n+1)^{n-1}$ reverse parking functions of length n.

Lecture 33
An exploration zonotopes, minkowski sums of segments. Zonotopes can be tiled with parallelepipeds. A brief introduction to hyperplane arrangements and fans.

Lecture 34
The normal fan of a zonotope is equal to the fan of the corresponding hyperplane arrangement. We continue exploring hyperplane arrangements and look specifically at the braid arrangement.

Lecture 35
Counting regions in hyperplane arrangements.

Lecture 36
The lattice of flats for a hyperplane arrangement is the intersection poset where elements are the intersections of hyperplanes in the arrangment ordered by reverse containment. We briefly explore the Möbius function of a poset and the characteristic polynomial of a hyperplane arrangement.

Lecture 37
More on the characteristic polynomial of a hyperplane arrangement. We explore the evaluation of the Möbius function for a flat in the lattice of flats of a hyperplane arrangement, and prove the deletion-contraction formula of the characteristic polynomial for a given hyperplane.

Lecture 38
Introduction to Möbius inversion. We explore the two-variable mobius function of a poset P and prove the Möbius inversion formula.

Lecture 39
We look at a number of examples of Möbius inversion formulas, including the poset of D_n - the divisors of n, the face poset of a polytope, and the poset of B_n - subsets of $[n]$.

Lecture 40
The finite field method for computing the characteristic polynomial of a hyperplane arrangement. If an arrangement A has all integer equations and q is a large enough prime then the characterstic polynomial evaluated at q is the number of points in $\mathbb{F}_q^n$ not on any hyperplane of A_q.

Lecture 41
Examples of characteristic polynomials for graphical arrangements. For a graph G, the characteristic polynomial of the corresponding graphical arrangement is equal to the chromatic polynomial of G.

Lecture 42
A summary of characteristic polynomials for finite reflection groups.

Lecture 43
We explore the Catalan arrangement.

Lecture 44
We explore the Shi arrangement.