matroid theory . 2007

final projects

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For the final project of the course, you will develop a solid understanding of a particular aspect of matroid theory that interests you. Your project must make a valuable contribution to the topic you are studying. You may, for instance:

- Understand the background and significance of an open problem in matroid theory, and solve it, or achieve some partial progress.
- Understand the current state of the art in a branch of matroid theory, and present it in a clear, concise, and useful survey. This must go beyond combining a couple of sources.
- Find a new way of thinking about or proving a known result.
- Write a computer program that will be useful to researchers in matroid theory.

due dates.

o Friday, April 20: rough outline. (1-2 pages)
o Monday, May 7: 2-3 preliminary report. (2-3 pages)
o Friday, May 25: final project. (10-20 pages in LaTeX, 11pt, single space)

some suggested topics.

These are some of the possible topics for a final project. I'll be happy to provide more information on any of these, and point you to some relevant literature. Please send me your suggestions to add to this list. Feel free to choose your own topic, but you'll need me to approve it before you start working on it. (I will need to approve your April 20 outline, but you'll probably want to talk to me before then.)

1. Different representations of linear matroids. A linear matroid can have various "essentially" different geometric representations. Study the reasons why some matroids have many different representations, while others have very few. For example, in [3], it is proved that the "tiling matroid" of Homework 3 has an essentially unique linear representation. Can you find a larger family of matroids for which this is true?

2. Different presentations of transversal matroids. Whitney described when different graphs give the same graphical matroid. Study the analogous question: When do different matching problems give the same transversal matroid? Start by studying Chapter 5 of [21]. Study how the duality between transversal and cotransversal matroids can be used to give different presentations of the same transversal matroid. Reference [2] may be useful.

3. Matroid polytopes and subdivisions. A matroid can be described beautifully in terms of its *matroid polytope*. Study the problem of tiling matroid polytopes with smaller matroid polytopes. This is a question which has recently appeared in several fields of mathematics, and deserves further study. This involves questions in algebra, optimization.

4. Matroids from tilings. In Homework 3 you saw how to build a matroid from the rhombus tilings of a triangle. In [3] we conjecture, and give evidence, that you can also build a matroid from certain tilings of a tetrahedron (or, more generally, a simplex in $\mathbb{R}^n$.) Understand these tilings and try to prove the conjecture.

5. Transversality. How do you determine whether a matroid is transversal? Start by studying Chapter 5 of [21]. Determine whether the above matroids, coming from three or n-dimensional tilings, are cotransversal. (They are in two dimensions. This is not known for any $n \geq 3$.)

6. A polynomial invariant of a matroid. (Nick Proudfoot.) Find a combinatorial interpretation for the polynomial $P_M(q)$, coming from the topology of hyperplane arrangements, which is defined by the equation

$$q^{r(M)} P_M(1/q) = \sum_{F \in L_M} P_{M|F}(q) \chi_{M/F}(q).$$

7. Ehrhart theory and matroids. Since matroids are characterized by the inequalities that their rank functions satisfy, they can be seen as the integer points inside a polytope. Study that polytope. What does Ehrhart theory tell us about the number of matroids? The number of linear matroids? References [9] and [11] may be relevant.

8. Mader matroids. Let $A_1, \ldots, A_n$ be disjoint sets of vertices of an undirected graph G; call $A_1 \cup \cdots \cup A_n$ the "terminals" of the graph. An A-path is one whose two ends are terminals in different A_is, and whose intermediate vertices are not terminals. An A-routing is a collection of vertex-disjoint A-paths. Say a set of vertices I is *independent* if there is an A-routing using every vertex in I (and possibly others) as a terminal. These independent sets form a *Mader matroid*. Study Mader matroids. Find a simple linear representation of them which will reprove the (Nov. 2006) theorem that Mader matroids are gammoids. [14] The ideas of [2] may be useful.

9. Valuated matroids. Given a weight vector w, suppose we want to find the minimum weight basis of a matroid. However, each basis B of the matroid is given an advantage f_B, so that its weight is $f_B + \sum_{b \in B} w_b$. If the given advantages are "nice" (they form a *valuated matroid*), the bases of minimum weight still form a matroid M_w. Study valuated matroids. Explain whether and how the greedy algorithm works in this setting. Learn about their applications in systems analysis [12] and/or tropical geometry [18].

10. Probability in hyperplane arrangements. Learn how the characteristic and Tutte polynomials are used in problems of network reliability. Explore open probabilistic questions about hyperplane arrangements, such as this one: If you draw n lines on a plane at random, and then delete each one of them with probability p, how many regions do you expect to see? [1, 19],[22, Chapter 6].

11. Face posets of nice arrangements. Study the poset of faces of nice hyperplane arrangements. For example, you could follow up on Felipe Rincón's work on the Shi arrangement:

$$x_i = x_j, \qquad x_i = x_j + 1 \qquad (1 \le i < j \le n)$$

    This could incorporate computer experimentation nicely.

12. Rank partition. Study the rank partition of a matroid, as explained in [16, Section 3.1.10].

13. Tutte polynomials of exceptional root systems. A root system is a highly symmetric collection of vectors satisfying certain geometrical properties, which appear in several fields of mathematics. Each root system defines a matroid; the Tutte polynomial of all but five of these matroids have been computed. [1] Understand what root systems are, and compute these remaining five polynomials. This project would be very intensive computationally.

14. Automorphisms of matroids. An automorphism of a matroid is a relabelling of the elements which preserves the matroid structure. These automorphisms form a group. Compute the automorphism groups of root system matroids. This has been done in some cases. [6]

15. Matroids and projective geometry. A *division ring* or *skew field* is a non-commutative analog of a field. Much of linear algebra generalizes to vector spaces over division rings. Study in detail the connection between matroid theory and projective geometry. In particular, explain these statements: A projective plane comes from a field if and only if it satisfies Pappus's theorem. A projective plane comes from a division ring if and only if it satisfied Desargues's theorem.

16. Basis exchange properties and White's conjecture Learn about the various forms of basis exchange that hold in a matroid, from a combinatorial and geometric point of view. Study White's 1980 conjecture that the toric ideal of a matroid is generated by quadrics. [20, Chapter 4].

17. Stanley's conjecture. Study Stanley's 1977 conjecture on the h-vector of a matroid. Understand Merino's proof for cographic matroids [10], and some of the known interpretations for this h-vector.

18. Lines in a hyperplane arrangement. A hyperplane arrangement $\mathcal{A}$ determines a linear matroid M. The lines that occur as intersections of hyperplanes in $\mathcal{A}$ determine a second matroid M'. You would explore the relationship between M and M'.

19. Matroid matrix-tree theorems. Understand the proof of Kirchhoff's matrix-tree theorem, and its generalization to linear matroids in [16, Exercise 11]. Apply this result to families of matroids with a nice combinatorial structure.

20. Regions determined by nice polytopes. Find the number of regions of the hyperplane arrangements determined by nice polytopes, such as a regular tetrahedron, cube, octahedron, dodecahedron, or icosahedron.

21. Infinite matroids. Study infinite matroids. [22, Chapter 3]. How do you determine whether two infinite matroids are isomorphic?

22. Matroids and rigidity. Learn about the connection between matroids and rigidity, and study the applications of matroids to engineering problems that involve rigidity of structures.[22, Chapter 1]

23. Matroids and topology. Learn about the very nice topological spaces associated to matroids, and how (some of) their topology can be described in terms of their combinatorics. [22, Chapter 7], [13]

24. Matroid optimization and algorithms. Study matroids from an optimization point of view, and the algorithmic questions that arise. [7, Chapter 11], [17, Part IV].

25. Matroids and knot theory. Explore the use of matroid theoretic invariants in problems in knot theory. In particular, explain how many of the knot polynomials studied in knot theory are closely related to the Tutte polynomial of matroid theory. Try to generalize some of the algebraic invariants of knot theory to matroids. [19]

References

[1] F. Ardila. *Computing the Tutte polynomial of a hyperplane arrangement.* To appear in the Pacific Journal of Mathematics. `http://math.sfsu.edu/federico/Articles/arrangem.pdf`

[2] F. Ardila. *Transversal and cotransversal matroids via their representations.* Electronic Journal of Combinatorics, 14 (2007), N6. `http://math.sfsu.edu/federico/Articles/lindstrom.pdf`

[3] F. Ardila. *Flag arrangements and triangulations of products of simplices.* To appear in Advances in Mathematics. `http://math.sfsu.edu/federico/Articles/flags.pdf`

[4] A. Borovik, I. Gelfand, and N. White. *Coxeter Matroids.* Birkhauser, 2003.

[5] A. Bjoner et al. *Oriented Matroids.* Cambridge University Press, 1993.

[6] L. Fern, G. Gordon, J. Leasure, and S. Pronchik. *Matroid automorphisms and symmetry groups.* Combinatorics, Probability and Computing 9 (2000), 105-123.

[7] R.L. Graham, M. Grtschel, and L. Lovsz. *Handbook of Combinatorics.* Elsevier, North Holland, 1995.

[8] L. Lafforgue. *Chirurgie des grassmanniennes.*, CRM Monograph Series, vol. 19, 2003.

[9] P. Lisonek, Combinatorial families enumerated by quasi-polynomials. Journal of Combinatorial Theory Ser. A 114 (2007), 619-630.

[10] C. Merino. *The chip firing game and matroid complex.* Discrete Mathematics and Theoretical Computer Science, Proceedings vol. AA, 2001, pp. 245-256.

[11] J. Morton, L. Pachter, A. Shiu, B. Sturmfels, and O. Wienand. *Convex rank tests and semigraphoids.* `http://arxiv.org/abs/math.CO/0702564`

[12] K. Murota. *Matrices and Matroids for Systems Analysis.* Algorithms and Combinatorics 20. Springer-Verlag, 1991.

[13] P. Orlik and H. Terao. *Arrangements of Hyperplanes.* Springer, Berlin, 1992.

[14] G. Pap. Mader matroids are gammoids. Preprint, 2006, `http://www.cs.elte.hu/egres/tr/egres-06-17.pdf`.

[15] M. Piff and D. Welsh. *On the Vector Representation of Matroids.* Journal of the London Mathematical Society 1970 s2-2(2):284-288

[16] V. Reiner. *Lectures on matroids and oriented matroids.* `http://www.math.umn.edu/~reiner/Talks/Vienna05/Lectures.pdf`

[17] A. Schrijver. *Combinatorial Optimization.* Springer, 2003.

[18] D. Speyer. *Tropical Linear Spaces.* `http://arxiv.org/abs/math.CO/0410455`

[19] D. Welsh. *Complexity: Knots, Colourings and Countings.* Cambridge University Press, 1993.

[20] N. White. *Theory of Matroids.* Cambridge University Press, 1986.

[21] N. White. *Combinatorial Geometries.* Cambridge University Press, 1987.

[22] N. White. *Matroid Applications.* Cambridge University Press, 1992.