(b). Prove that $|\chi_G(-1)|$ is the number of ways of orienting each edge of G in such a way that no directed cycles are formed

10/10

Pf. Assign a basis vector e_i to each vertex in G, and define $H_{i,j}$ to be the hyperplane $x_i = x_j$. We know from class that $|\chi_G(-1)|$ counts the number of regions for this hyperplane arrangement, so it suffices to find a bijective mapping between the regions of this hyperplane arrangement and the acyclic orientations of G. Fix a region in the hyperplane arrangement, and a particular hyperplane $H_{i,j}$. This hyperplane corresponds to an edge $\{v_i, v_j\}$ in G. Orient $\{v_i, v_j\}$ as follows: If the region we chose is contained in the inequality $x_i < x_j$, then direct the edge from v_j to v_i. If it lies in $x_i > x_j$, orient the edge from v_i to v_j. WTS this is acyclic. Suppose there is a cycle $v_1, \ldots, v_n$. Then this region satisfies $x_1 < x_2 < \cdots < x_n < x_1$, which is impossible, so it is acyclic. Next, fix an acyclic region. WTF a point in one of the regions uniquely determined by the orientation. WT assign a number 1 to n to each vertex $v_1, \ldots, v_n$. Find all the t_1 sinks in the graph. We know at least 1 exists, since the orientation is acyclic. Assign them $1, 2, \ldots, t_1$. Then remove the sinks from the graph, and the edges going to them. Find the t_2 new sinks, and label them $t_1+1, \ldots, t_1+t_2$. Continue in this manner until all n vertices have a number between 1 and n. This point we have found lies in the region that we initially mapped to this orientation, so we have a bijection and we are done.