5 (a) Claim: The number of ways to properly color a loopless graph G with q colors is $q^c \chi_L(q)$, where $c = \#$ of connected components of G and χ_L is the characteristic polynomial of the lattice of flats of G.

Let g and f be functions on the lattice of flats L of G, defined as follows:

$g(X) =$ the number of ways to color G such that the flat X is comprised of monochromatic edges.

$f(X) =$ the number of ways to color G such that the flat X is comprised of the <u>only</u> monochromatic edges in the coloring.

Claim 1: $g(X) = q^c \cdot q^{r - r(X)}$ where $c = \#$ of connected components of G and r is the rank function on L.

If X must consist of monochromatic edges, then where do we get freedom to color arbitrarily? Well, each connected component of X must be a single color, by adjacency of monochromatic edges. So we have q choices for what to color each connected component of X. Then, whatever vertices are leftover in $G \setminus X$ can each be colored in q different ways. Let $c_X = \#$ of connected components of X, V_X be the number of vertices in X, and V be the total number of vertices in G.

Then $g(X) = q^{c_X + V - V_X}$. We would like for $c_X + V - V_X = c + r - r(X)$. Indeed, we know that in a forest the # of vertices $=$ # of edges $+$ # of connected components. So for a spanning forest of G, we have $V = r + c$. Similarly, $V_X = r(X) + c_X$. Substituting these identities into the previous expression gives us $c_X + V - V_X = c + r - r(X)$ as desired.

Claim 2: $g(X) = \sum_{X \leq Y} f(Y)$.

Since flats are defined to be the closed sets, and adding any new edge to a flat will introduce a new vertex, we may think of a flat as being the "most complete" possible subgraph of G for a given set of vertices. In that sense it makes sense to characterize monochromatic components as flats. Indeed, if a connected subset of the vertices of G is all the same color, then all possible edges between these vertices are monochromatic edges. This set of edges defines a flat.

5 (a) (continued)

That being said, the result of the claim follows almost immediately. Indeed, If we have a coloring of G such that the edges of X are monochromatic, then the set of <u>all</u> monochromatic edges of this coloring forms a flat of G which contains X. Call this flat Y. The number of all such colorings is precisely $f(Y)$. In a coloring where the set of monochromatic edges does <u>not</u> equal Y, we will be looking at some <u>other</u> flat Z which also contains X. So we are not double counting, and we are covering all possibilities. Hence the claim is true.

Finally, by the upside-down Möbius inversion formula, we have

$$f(X) = \sum_{X \subseteq Y} \mu(X,Y)\, g(Y) = \sum_{X \subseteq Y} \mu(X,Y)\, q^c \cdot q^{r-r(Y)} = q^c \sum_{X \subseteq Y} \mu(X,Y)\, q^{r-r(Y)}.$$

Letting $X = \emptyset$, we conclude that

the number of ways to properly color G with q colors = the number of ways to color G such that the set of monochromatic edges is empty $= f(\emptyset) = q^c \sum_{\emptyset \subseteq Y} \mu(\emptyset, Y)\, q^{r-r(Y)} = q^c \sum_{Y \in L} \mu(Y)\, q^{r-r(Y)} = q^c \chi_L(q)$.