**5**(a) To begin, we first establish a lemma that for any lattice $L$ whose set of atoms is $A$, and any $x \in L$, we have that $\mu_L(x)$ is equal to the number $\mu'(x)$ defined by

$$\mu'(x) = \sum_{\substack{S \subseteq A \\ \bigvee S = x}} (-1)^{|S|}, \tag{3}$$

where $\bigvee S$ denotes the join of all atoms of $S$. To check this we must verify that the sums defining $\mu_L$ hold of it. First, we have $\mu'(\hat{0}) = 1$ since $\hat{0}$ is a join of a set of atoms in only one way, namely as the join of the empty set. Otherwise, let $x \geq \hat{0}$ and let $B$ be the set of atoms $a$ with $a \leq x$. Then, for a set $S$ of atoms, we have $\bigvee S \leq x$ iff $S \subseteq B$; 'if' is true by the definition of the join, while 'only if' is true because if $a \not\leq x$ then no join involving $a$ can be $\leq x$, by transitivity. Therefore we have

$$\sum_{x' \leq x} \mu'(x) = \sum_{S \subseteq B} (-1)^{|S|} = 0$$

because $B$ is nonempty. This proves the lemma.

Now, we'll prove that not $\chi_M(q)$ but $q\chi_M(q)$ is the number of $q$-colorings of $G$. We count the $q$-colorings of $G$ using the inclusion-exclusion[1] principle on the edges, which are the atoms of $L_M$. For any set $S$ of edges of $G$, let $n(S)$ be the number of colorings of $G$ such that each edge of $S$ receives the same color at both ends. Then, by the inclusion-exclusion principle, the number $n$ of colorings of $G$ with $q$ colors and no edge colored similarly at each end is

$$n = \sum_{S \subseteq E(G)} (-1)^{|S|} n(S).$$

[1]Some, perhaps all, of my uses of inclusion-exclusion should be thought of as *really* uses of Möbius inversion on lattices other than $2^{[n]}$. They're the same thing anyway, by my lemma.

Now, for some set of edges $S$, a coloring $C$ of $G$ has the property that each edge in $S$ has the same color at each end iff each connected component of the subgraph $H$ of $G$ consisting of edges in $S$ receives a single color; so $n(S)$ is $q$ to the power of the number of components of $H$ (including components with just one vertex and no edges). But the number of such components is precisely $|V(G)| - r(S) = r(M) + 1 - r(S)$, as we can see by a simple induction: the empty subgraph of $G$ has $|V(G)|$ components, and adding each edge in some spanning forest of $H$ removes one component. Therefore, by our lemma,

$$n = \sum_{S \subseteq E(G)} (-1)^S q^{r(M)+1-r(S)} = \sum_{x \in L_M} q^{r(M)+1-r(X)} = q\chi_M(q).$$

(b) We begin by again using (3), by which

$$\begin{aligned}
\chi_M(-1) &= \sum_{x \in L(M)} \mu_M(x)(-1)^{r(M)-r(x)} \\
&= (-1)^{r(M)} \sum_{x \in L(M)} \mu_M(x)(-1)^{r(x)} \\
&= (-1)^{r(M)} \sum_{x \in L(M)} (-1)^{r(x)} \sum_{\substack{H \subseteq G \\ \mathrm{cl}\, H = x}} (-1)^{|H|} \\
&= (-1)^{r(M)} \sum_{H \subseteq G} (-1)^{|H|+r(H)} \\
&= (-1)^{r(M)} \sum_{H \subseteq G} (-1)^{|H|-r(H)}.
\end{aligned}$$

Here, as everywhere in this problem, we demand that a subgraph of $G$ (or of an arbitrary graph) contain all vertices of $G$, and identify subgraphs of $G$ with their sets of edges. So our eventual aim will be to show that this sum $\sum_{H \subseteq G} (-1)^{|H|-r(H)}$ counts the acyclic orientations of $G$. Let $N$ denote the number of such orientations.

Using inclusion-exclusion, we may count the acyclic orientations of $G$ in terms of the orientations containing particular simple (i.e. non-self-intersecting) directed cycles of $G$. Let $C$ be the set of all directed cycles that appear in any orientation of $G$; then

$$N = \sum_{S \subseteq G} (-1)^{|S|} N(S) \tag{4}$$

where $N(S)$ is the number of orientations of $G$ which contain every directed cycle in $S$.

When $S$ contains two cycles assigning opposite directions to a single edge, then certainly $N(S) = 0$. If this is not the case we'll call $S$ *compatible*. In an orientation of $G$ containing a compatible set $S$ of cycles, the direction of all edges in $\bigcup S$ are fixed while the other edges are unrestricted, so that writing $\operatorname{supp} S$ for the subgraph of $G$ containing just those edges which receive a direction in $S$, we have $N(S) = 2^{|E(G)\setminus\bigcup S|}$. This also counts the subgraphs of $G$ that contain the undirected subgraph $\bigcup S$, so that we can rewrite the sum (4) as

$$\begin{aligned} N &= \sum_{S\subseteq C \text{ compatible}} \sum_{\operatorname{supp} S\subseteq H\subseteq G} (-1)^{|S|} \\ &= \sum_{H\subseteq G} \sum_{\substack{S\subseteq C \text{ compatible} \\ \operatorname{supp} S\subseteq H}} (-1)^{|S|}. \end{aligned} \tag{5}$$

Pick a subgraph $H$ of $G$. The term of this last sum corresponding to $H$ is counting the compatible sets of directed cycles $S$ whose union is supported on $H$, weighted by $(-1)^{|S|}$. We claim that we may reinterpret this as counting the oriented subgraphs $K$ of $H$ which are a union of directed cycles in some way, weighted by $(-1)^{|K|-r(K)}$. Equivalently, if we collect the sets $S$ together by the form of the graph $K = \bigcup S$, then our claim is that the number of $S$ that give rise to a given $K$, weighted by $(-1)^{|S|}$, is $(-1)^{|K|-r(K)}$.

For this, we fix $K$ and consider the subgraphs $L$ of $K$ that are unions of cycles. These form a partial order $\mathcal{L}$ by inclusion, which indeed is a lattice: the union of two subgraphs $L$, $L'$ that are unions of cycles is itself a union of cycles, so it's the join $L \vee L'$, and then since $\mathcal{L}$ is finite and has a minimal element it also has meets (the meet of $L$ and $L'$ is the join of all their mutual lower bounds). Furthermore, the atoms of $\mathcal{L}$ are just the single directed cycles contained in $K$, so by definition $\mathcal{L}$ is atomic. Therefore our lemma from (a) applies here, telling us that the number of $S$ that give rise to $K$ is simply $\mu_{\mathcal{L}}(K)$. So we want to show that

$$\mu\mathcal{L}(K) = (-1)^{|\operatorname{supp} K|-r(\operatorname{supp} K)}. \tag{6}$$

This exponent $|\operatorname{supp} K| - r(\operatorname{supp} K)$ gives the size of the complement of any spanning forest of $\operatorname{supp} K$.

We first argue that $|\operatorname{supp} K| - r(\operatorname{supp} K)$ is the rank of $\mathcal{L}$, and that $\mathcal{L}$ is graded by the function $r_{\mathcal{L}} : L \mapsto |\operatorname{supp} L| - r(\operatorname{supp} L)$. Suppose $L$ covers $L'$. We claim that $L \backslash L'$ is a set of $k$ edges $(v_{i-1}, w_i)$, where $c_0, c_1, \ldots, c_{k-1}, c_k = c_0$ are distinct weakly (and therefore strongly) connected components of $L'$, and $v_i, w_i \in c_i$. From this description one can check $r_{\mathcal{L}}(L) = r_{\mathcal{L}}(L') + 1$: moving from $L'$ to $L$ introduces $k$ edges and the spanning forest requires expansion by $k - 1$ edges.

To see this, take any edge $e$ of $L \setminus L'$. Then $L \setminus L'$ contains every edge $e'$ such that every cycle containing $e'$ contains $e$ as well, because $L'$ is a union of cycles. Removing only these edges $e'$ leaves a graph that is a union of cycles, so since $L$ covers $L'$ the difference contains only these edges $e'$. Now, $L'$ is partitioned into strongly connected components. Consider the graph obtained by contracting along every edge of $L'$; its vertices corresponds to components of $L'$, the only edges that remain are those of $L \setminus L'$, and every cycle in $L$ contracts to a cycle in $L'$. Then this graph is nonempty, since it contains $e$; it contains a cycle, because $e$ is contained in some cycle; and it cannot contain more than a single cycle, because if it did a single cycle within it would come from a proper subgraph of $L$ containing $L'$ strictly, contradicting covering.

Now, having established the rank function on $\mathcal{L}$, we're ready to attack (6). We'll induce on the rank of $\mathcal{L}$. We'll take as base cases $\operatorname{rank} \mathcal{L} = 0, 1$; these are trivial to check. So suppose $\mathcal{L}$ has rank $\geq 2$. By the inductive hypothesis applied to every interval $[\hat{0}, x]$ of $\mathcal{L}$, the Möbius function of $\mathcal{L}$ is $\mu_{\mathcal{L}}(x) = (-1)^{r_{\mathcal{L}}(x)}$ everywhere except possibly at $x = \hat{1} = L$. So we need to check that this holds there as well. In other words, if we define $\nu : \mathcal{L} \to \mathbb{Z}$ to agree with $\mu_{\mathcal{L}}$ away from $L$ and to have $\nu(L) = (-1)^{r_{\mathcal{L}}(L)}$, we want to check that $\sum_{x \in L} \nu(x) = 0$.

Let $x \in \mathcal{L}$, and let $M$ be the graph obtained from $L$ by contracting every edge of $x$. Then unions of cycles in $M$ correspond naturally via an order-preserving bijection to unions of cycles in $L$ that contain $x$. This establishes that the interval $[x, \hat{1}]$ of $\mathcal{L}$ has the same form as $\mathcal{L}$ (i.e. it's the lattice of a cycle arrangement), so that if $x \neq \hat{0}$ then the inductive hypothesis applies to it. In particular, its Möbius function is $\nu$, up to a sign. Now, let $L'$ be a coatom of $\mathcal{L}$, so that we have $\sum_{0 \leq x \leq L'} \nu_(x) = 0$ since $\nu$ is simply the Möbius function on $[\hat{0}, x]$. The poset $\mathcal{L} \setminus [\hat{0}, x]$ contains one minimal element corresponding to each atom (i.e. cycles) in $L$ not less than $L'$; let $A$ be the set of these atoms. By inclusion-exclusion, we have

$$\sum_{y \not\leq x} \nu(y) = \sum_{S \subseteq A} (-1)^{|S|} \sum_{y \geq \bigvee S} \nu(y) = \sum_{S \subseteq A} (-1)^{|S|} \sum_{y \geq \bigvee S} \mu_{[y,\hat{1}]}(y) \tag{7}$$

in which each term of the outer sum on the right will be 0 so long as no join of elements of $A$ is $\hat{1}$, and certainly if the join of all elements of $A$ is not $\hat{1}$. For this we invoke the characterisation of coatoms we came to in our discussion of the rank function, giving $L \setminus L'$ as a collection of edges of some particular form. The elements of $A$ are all the (simple) cycles containing $L \setminus L'$. But if any edge $e \in L \setminus L'$ is incident to a vertex $v$ to which another edge $e' \notin L \setminus L'$ is also incident in the same sense (either both are in-edges or both are out-edges), then $e'$ can be contained in no cycle containing $e$, so $\bigvee A$ doesn't contain $e'$; whereas if this doesn't happen, then $L \setminus L'$ is a cycle and a component of $L$ unto itself, and this cycle is the only element of $A$, and then $\bigvee A \neq L$ since $L'$ is nonempty. So the value in (7) is 0, proving our claim about $\mu_{\mathcal{L}}$ in (6).

Stepping back, what we've just justified gives us the equality

$$\sum_{\substack{S \subseteq C \text{ compatible} \\ \operatorname{supp} S \subseteq H}} (-1)^{|S|} = \sum_{\substack{K \text{ is a union of directed cycles} \\ \operatorname{supp} K \subseteq H}} (-1)^{|\operatorname{supp} K| - r(\operatorname{supp} K)}$$

which we were planning to use to get a handle on (**??**). We now claim that this sum is $(-1)^{|H|-r(H)}$. This, by our very first chain of equalities, will give us $|\chi_M(-1)| = N$, finishing the problem.

For this, we use induction on the number of vertices of $H$ (over arbitrary $H$; $H$ is no longer constrained to be a subgraph of any particular graph $G$). As a base case, suppose that all edges of $H$ are loops; this is in particular necessarily true when $H$ has just one vertex. Then each directed simple cycle on $H$ consists of just one of these loops, given either direction, and every loop yields two directed simple cycles in this way. Therefore every *partial orientation* $K$ of $H$, i.e. an assignment of directions to some of the edges of $H$, is a union of directed cycles. The support of such a partial orientation $K$ simply contains all the loops which are directed in either sense, and the rank of this support is always 0, so the weight with which $K$ is counted is $(-1)^{|\operatorname{supp} K|}$. Therefore our sum

$$\sum_{K : \operatorname{supp} K \subseteq H} (-1)^{H - r(H)}$$

(over all $K$ that are unions of directed cycles) factors into a product of one sum for each loop of $H$, of the form $1+(-1)+(-1)=-1$, the 1 arising from leaving this loop undirected and the two $-1$s from giving it either of the two possible directions. The sum thus evaluates to $(-1)^{|H|} = (-1)^{|H|-r(H)}$, as desired.

For the inductive step, we assume $H$ contains a nonloop $e$. Let $H'$ be the contraction of $H$ on $e$. Then $|H'| = |H| - 1$ and $r(H') = r(H) - 1$, so that $(-1)^{|H'|-r(H')} = (-1)^{|H|+r(H)}$. We'll then proceed by setting up a correspondence between unions $K$ of directed cycles on $H$ and unions $K'$ of directed cycles on $H'$. It won't be a one-to-one correspondence; but it will have the property that each $K$ corresponds to just one $K'$, and the sum of the weights $(-1)^{|\operatorname{supp} K|-r(\operatorname{supp} K)}$ over all $K$ corresponding to a given $K'$ is the same weight for $K'$. This, together with the observation $(-1)^{|H'|-r(H')} = (-1)^{|H|+r(H)}$, will complete the induction.

So, let $K$ be a union of directed cycles on $H$. Let $K'$ be the natural corresponding object on $H'$, obtained by leaving all the edges other than $e$ alone. This is still a union of cycles. Conversely, given $K'$, we wish to determine which $K$ it could have come from. There are at most three possibilities for $K$: the new edge $e$ can be undirected or directed in either of two senses, and the remainder of the partial orientation is determined by $K'$. Let $v_1 \neq v_2$ be the endpoints of $e$ in $H$, and let $v$ be the vertex they merge to in $H'$. Then we have a number of cases to consider, according to what the in- and out-degrees of $v_1, v_2$ are in $K'$ (i.e. what these degrees become in $K$ if $e$ is chosen to be undirected).

Since $v$ is in a union of cycles, it has an in-edge if and only if it has an out-edge. Thus we have the following possibilities for the in- and out-degrees of $v_1$ and $v_2$:

**Some $v_i$ has no incident directed edges.** In this case $e$ must be made undirected in $K$ (i.e. there is one possible $K$). Then $|\operatorname{supp} K| = |\operatorname{supp} K'|$ and $r(\operatorname{supp} K) = r(\operatorname{supp} K')$, so weight $K$ = weight $K'$.

**Some $v_i$ has in-edges but no out-edges, or vice versa.** First note that if one of $v_1$ or $v_2$ fulfil this description then it's with respect to opposite directions, and also that this case and the last cannot both occur: this is because $v$ must have an in-edge iff it has an out-edge. Therefore, in this case, $e$ must be made directed in $K$ in order that $v_i$ have both an in- and an out-edge, and there is a unique valid choice of direction.

We have $|\operatorname{supp} K| = |\operatorname{supp} K'| + 1$ and $r(\operatorname{supp} K) = r(\operatorname{supp} K') + 1$ (a spanning forest for $\operatorname{supp} K'$ needs to be supplemented with $e$ to get one for $\operatorname{supp} K$), so again $\operatorname{weight} K = \operatorname{weight} K'$.

**Each $v_i$ has both in- and out-edges.** In this case all three possibilities for $K$ can be checked to yield a union of cycles. Let $K_0, K_1, K_2$ respectively denote the partial orientations where $e$ is undirected, directed $v_1 \to v_2$, and directed $v_2 \to v_1$. Now, for $K_0$ we have $|\operatorname{supp} K_0| = |\operatorname{supp} K'|$ but $r(\operatorname{supp} K_0) = r(\operatorname{supp} K) + 1$ (pulling a vertex apart will disconnect the spanning tree, and it needs another edge to be reconnected), so that $\operatorname{weight} K_0 = -\operatorname{weight} K$. However, $K_1$ and $K_2$ have $\operatorname{weight}(K_1) = \operatorname{weight}(K_2) = \operatorname{weight}(K)$ just as in the previous case, so that

$$\operatorname{weight}(K_0) + \operatorname{weight}(K_1) + \operatorname{weight}(K_2) = (-1+1+1)\operatorname{weight}(K) = \operatorname{weight}(K).$$

Thus in every case we've confirmed the required properties of our correspondence with respect to weight. As remarked above this completes our final induction, and thereby completes the proof.