Gödel's completeness and matroid representability

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Here is a nice application of Gödel's completeness theorem to matroid representability.

Theorem. *If a matroid M is representable over a field of characteristic p for arbitrarily large primes p, then it is representable over a field of characteristic zero.*

Proof. We will work on the language of rings $\mathcal{L} = \{+, \cdot, 0, 1\}$. Denote by FT the theory consisting of all the field axioms. For $n \in \mathbb{N}$ let ϕ_n be the formula

$$\underbrace{1 + 1 + \cdots + 1}_{n \text{ times}} = 0.$$

The theory FT_0 of fields of characteristic 0 is just $FT \cup \{\neg\phi_1, \neg\phi_2, \neg\phi_3, \dots\}$.

Let M be a matroid. Notice that there is a formula ψ_M such that for every field F, $F \models \psi_M$ if and only if M is representable over F (ψ_M could be the list of all the bases and circuits of M). For example, for the uniform matroid $U_{2,3}$ this formula would be

$$\exists\ a_1, a_2, b_1, b_2, c_1, c_2 \text{ such that } \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \text{ and } \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \text{ are linearly dependent,}$$
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$$\text{and } \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \text{ and } \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \text{ are linearly independent}$$

(since $U_{2,3}$ has rank 2, we can assume that just 2 coordinates are needed for the representation).

Now assume by contradiction that M is not representable over any field of characteristic 0, that is, $FT_0 \models \neg\psi_M$. Then, by Gödel's completeness theorem, we have that $FT_0 \vdash \neg\psi_M$. Since deductions are finite, this deduction of $\neg\psi_M$ uses only finitely many of the axioms in FT_0. Therefore, there exists an $N \in \mathbb{N}$ such that $FT \cup \{\neg\phi_1, \neg\phi_2, \neg\phi_3, \dots, \neg\phi_N\} \vdash \neg\psi_M$. But then $FT \cup \{\neg\phi_1, \neg\phi_2, \neg\phi_3, \dots, \neg\phi_N\} \models \neg\psi_M$, that is, M is not representable over any field of characteristic greater than N. $\square$