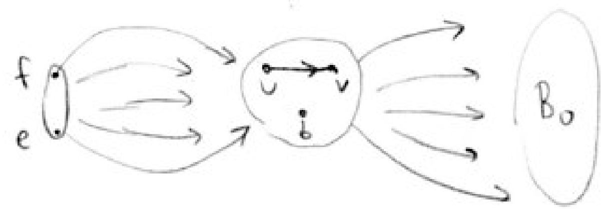
Claim. If G contains no routing from $\{e,f\}$ to B_0, then is a "bottleneck vertex" b such that any path from e or f to B_0 passes through b.

Proof. Consider a counterexample with the minimum number of edges possible.

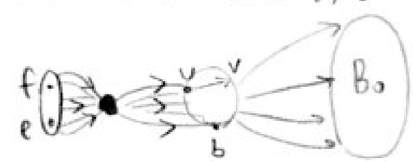
Let $\underset{u}{\bullet}\!\!\rightarrow\!\!\underset{v}{\bullet}$ be an edge and $H = G - (\underset{u}{\bullet}\!\!\rightarrow\!\!\underset{v}{\bullet})$. There is still no routing from e,f to B_0 in H. So let b be a bottleneck from $\{e,f\}$ to B_0 in H. $\Rightarrow \{b,u,v\}$ bottleneck for $\{e,f\} \to B_0$ in the graph G.



If I had a routing $\{e,f\} \to \{u,b\}$ and a routing $\{b,v\} \to B_0$, I would get a routing $\{e,f\} \to B_0$

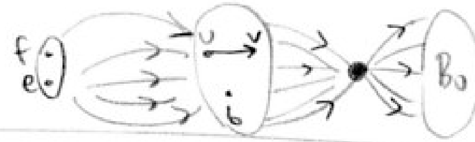
So either

- I have no routing $\{e,f\} \to \{u,b\}$ $\xRightarrow{\text{minimality of } G}$ bottleneck from $\{e,f\} \to \{u,b\}$



or

- I have no routing $\{b,v\} \to B_0 \Rightarrow$ bottleneck from $\{b,v\} \to B_0$



■

Note. More generally:

Menger's Theorem. $G=(V,E)$ directed graph, $A,B \subseteq V$:
$$\begin{pmatrix}\text{size of largest routing} \\ \text{from } A \text{ to } B\end{pmatrix} = \begin{pmatrix}\text{size of smallest bottleneck} \\ \text{from } A \text{ to } B\end{pmatrix}$$

Pf. Same. ■