4 Show that if e and f are parallel elements in a cotransversal matroid M, then $M \setminus e$ is also cotransversal.

Suppose e and f are parallel. Then they each belong to a basis, but not the same basis.

Claim 1: All routes from e and f to B_0 pass through a bottleneck.

[First, what is a bottleneck? It is a vertex in the digraph through which every route from e to B_0 and every route from f to B_0 must pass.

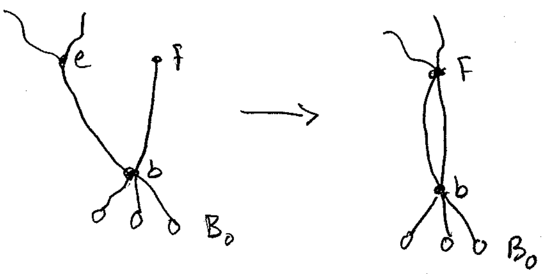
Call this vertex b. Suppose there is no such bottleneck. Then there exist disjoint routes from f and e to $b_f, b_e \in B_0$, respectively. But then $\{e, f, B_0 - (b_e \cup b_f)\}$ is a basis containing e and f. ⨳]

Now to show that $M \setminus e$ is cotransversal, we will exhibit a rank function on $E-e$ that matches the rank function on E. First, let's prove one more claim:

Claim 1′: A vertex, all of whose routes to B_0 pass through b, is parallel to f.

[Let x be the vertex in question. Take the rest of B_0 to extend to a basis. Route x to b and then to B_0. So x belongs to a basis. But x and f clearly can't belong to the same basis, because their routes to B_0 must intersect at b. So x and f are parallel elements].

Now let us modify the digraph by identifying e and f. Notice that any vertices between e and b or between f and b were parallel to e, f, and each other by Claim 1′. By identifying e and f we preserve this parallelism. I claim that the rank function on this new digraph M' is the same as the rank function on $M \setminus e$.



Let X be a subset of vertices of $M-e$. Then $r_{M \setminus e}(X)$ is the maximum number of elements of X which can be simultaneously, disjointly routed to B_0. Call this set X'. The only way that the routes of X' in $M \setminus e$ can differ from routes of X' in M' is if any route passes through e. If so, then no other route of X' can pass through any parallel element to e. Hence we can replace this element of X' with f, and we will still have a routing to the same subset of B_0, which now avoids e. This describes a maximal routing of X' in M' with the same size. Hence there is a 1-1 correspondence between independent sets of $M \setminus e$ and M', where rank is preserved. So $M \setminus e$ is isomorphic to the cotransversal matroid M'.